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Representations of centrally-extended Lie algebras over differential operators and vertex algebras

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Abstract

We construct irreducible modules of centrally-extended classical Lie algebras over left ideals of the algebra of differential operators on the circle, through certain irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries. The structures of vertex algebras associated with the vacuum representations of these algebras are determined. Moreover, we prove that under certain conditions, the highest-weight irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries naturally give rise to the irreducible modules of the simple quotients of these vertex algebras. From vertex algebra and its representation point of view, our results with positive integral central charge are high-order differential operator analogues of the well-known WZW models in conformal field theory associated with affine Kac–Moody algebras. Indeed, when the left ideals are the algebra of differential operators, our Lie algebras do contain affine Kac–Moody algebras as subalgebras and our results restricted on them are exactly the representation contents in WZW models. Similar results with negative central charge are also obtained.

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1. Introduction

The well-known $W_{1+\infty}$ Lie algebra is the centrally-extended Lie algebra of differential operators on the circle. It serves as the symmetry algebra of the famous KP-hierarchy in integrable systems. In fact, it is the Lie algebra of both rank-one charged quadratic free bosonic fields and rank-one charged quadratic fermionic fields. Kac and Radul [23] gave a classification of quasifinite highest-weight irreducible modules of the $W_{1+\infty}$ algebra. Frenkel, Kac, Radul and Wang [13] proved that the categories of irreducible modules of the vertex operator algebras associated to $\mathcal{W}_{1+\infty}$ and $\mathcal{W}(gl_N)$ with positive integral central charge N are equivalent. Moreover, Kac and Radul [24] used free quadratic bosonic fields to study the irreducible representations of the vertex operator algebras associated with $\mathcal{W}_{1+\infty}$ of negative integral central charge. The results in [23] were extended to certain Lie subalgebras of $\mathcal{W}_{1+\infty}$ by Kac, Wang and Yan [26].

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In general, the Lie algebras of charged free quadratic bosonic fields and fermionic fields of any rank are centrally-extended general linear Lie algebras over differential operators on the circle. They are the symmetry algebras of the multi-component KP-hierarchies in integrable systems (cf. [30]). Boyallian and Liberati [5] studied certain classical Lie subalgebras of the Lie algebra of matrix differential operators on the circle. Based on our work [35], Ma [28, 29] systematically investigated the conformal algebra structures and two-cocycles of centrally-extended classical Lie superalgebras over left ideals of the algebra of differential operators on the circle. Borcherds' notion of vertex algebra in [4] was proved by us [36] to be equivalent to that of the Hamiltonian operator via conformal algebra. For the other closely related works, the reader may refer to [1–3,6–8].

In this paper, we give constructions of irreducible modules of centrally-extended classical Lie algebras over left ideals of the algebra of differential operators on the circle, through certain irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries. The structures of vertex algebras associated with the vacuum representations of these algebras are determined. Moreover, we prove that under certain conditions, the highest-weight irreducible modules of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries naturally give rise to the irreducible modules of the simple quotients of these vertex algebras. It turns out that our results with positive integral central charge resemble those for the vertex operator algebras associated with affine Kac–Moody algebras, which are called *WZW models* in conformal field theory. This leads to a fundamental difference between our work and those of Kac et al. Similar results with negative central charge are also obtained. Bearing in mind, special examples of the Hecke algebras and group algebras, we present general constructions of representations of certain Lie subalgebras of the centrally-extended Lie algebra of the tensor product of any associative algebra with the algebra of differential operators on the circle. In the case of group algebras, this gives an important connection between group representations and Lie algebra representations, by which we obtain that any simple group can be an automorphism subgroup of a certain vertex algebra (this is a certain generalization of the well-known moonshine representation of the Monster group). When the associative algebra is a Hecke algebra, our work gives a link between the Hecke algebras and vertex algebras, which is a connection people naturally expect. Our representation theory can be viewed as quadratic generalizations of the free field theory. Below we give a more detailed and technical introduction.

Throughout this paper, all the variables are formal and commute with each other. All the vector spaces are assumed over $\mathbb{C}$, the field of complex numbers. Denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N}$ the additive semigroup of nonnegative integers.

Denote $\partial_t = \mathrm{d}/\mathrm{d}t$. Let

$$\mathbb{A} = \sum_{i=0}^{\infty} \mathbb{C}[t, t^{-1}]\partial_t^i \tag{1.1}$$

be the algebra of differential operators on the circle. Let $M_{n\times n}(\mathbb{A})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{A}$. Denote by $E_{i,j}$ the $n \times n$ matrix with 1 as its (i, j)-entry and 0 as the others. Define the vector space $\widehat{gl}(n, \mathbb{A}) = M_{n\times n}(\mathbb{A}) \oplus \mathbb{C}\kappa$ and its Lie bracket:

$$\begin{aligned}[t^{m_1}\partial_t^{r_1} E_{i_1,j_1} + \mu_1\kappa, t^{m_2}\partial_t^{r_2} E_{i_2,j_2} + \mu_1\kappa] &= \delta_{j_1,i_2} t^{m_1}\partial_t^{r_1} \cdot t^{m_2}\partial_t^{r_2} E_{i_1,j_2} - \delta_{i_1,j_2} t^{m_2}\partial_t^{r_2} \cdot t^{m_1}\partial_t^{r_1} E_{i_2,j_1} \\ &\quad + (-1)^{r_1}\delta_{i_1,j_2}\delta_{j_1,i_2}\delta_{r_1+r_2,m_1+m_2} r_1!r_2! \binom{m_1}{r_1+r_2+1}\kappa \end{aligned} \tag{1.2}$$

for $i, j \in \overline{1, n}$ and $m_1, m_2, r_1, r_2 \in \mathbb{N}$. Fix an element $\vec{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$. The subspace

$$\widehat{gl}(\vec{\ell}, \mathbb{A}) = \sum_{i,j=1}^{n} \mathbb{A}\partial_t^{\ell_j} E_{i,j} + \mathbb{C}\kappa \tag{1.3}$$

of $\widehat{gl}(n, \mathbb{A})$ forms a Lie subalgebra. The well-known Lie algebra $W_{1+\infty}$ is the special case of $\widehat{gl}(\vec{\ell}, \mathbb{A})$ when $n = 1$ and $\vec{\ell} = 0$.

Denote $\overline{1, n} = \{1, 2, \ldots, n\}$, and set $i^* = n + 1 - i$ for $i \in \overline{1, n}$. We fix $\epsilon \in \{0, 1\}$ and take

$$\vec{\ell} = (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{N}^n \quad \text{such that } \{\ell_1, \ell_2, \ldots, \ell_n\} \subset 2\mathbb{N} + \epsilon \tag{1.4}$$

and $\ell_i = \ell_{i^*}$ for $i \in \overline{1,n}$. The subspace

$$\hat{o}(\vec{\ell},\mathbb{A}) = \sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\sum_{m\in\mathbb{Z}}\mathbb{C}(t^m\partial_t^{r+\ell_j}E_{i,j} - (-1)^{\epsilon}(-\partial)^r t^m \partial_t^{\ell_i}E_{j^*,i^*}) + \mathbb{C}\kappa \tag{1.5}$$

of $\widehat{gl}(n,\mathbb{A})$ forms a Lie subalgebra. Next we suppose that $n = 2n_0$ is an even positive integer. Moreover, we define the parity of indices:

$$p(i) = 0, \qquad p(n_0+i) = 1 \quad \text{for } i \in \overline{1,n_0}. \tag{1.6}$$

The subspace

$$\widehat{sp}(\vec{\ell},\mathbb{A}) = \sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\sum_{m\in\mathbb{Z}}\mathbb{C}(t^m\partial_t^{r+\ell_j}E_{i,j} - (-1)^{p(i)+p(j)+\epsilon}(-\partial_t)^r t^m \partial_t^{\ell_i}E_{j^*,i^*}) + \mathbb{C}\kappa \tag{1.7}$$

of $\widehat{gl}(n,\mathbb{A})$ forms a Lie subalgebra. The element κ corresponds to the *central charge* in physics.

Note that the Lie algebras $\hat{o}(\vec{\ell},\mathbb{A})$ and $\widehat{sp}(\vec{\ell},\mathbb{A})$ are in general not graded by conformal weights. One of the main objectives in this paper is to construct irreducible modules of $\widehat{gl}(\vec{\ell},\mathbb{A})$, $\hat{o}(\vec{\ell},\mathbb{A})$ and $\widehat{sp}(\vec{\ell},\mathbb{A})$ through weighted irreducible modules (may not be the highest-weight type) of centrally-extended classical Lie algebras of infinite matrices with finite number of nonzero entries.

For $i, j \in \overline{1,n}$ and $r \in \mathbb{N}$, we denote

$$E_{i,j}(r,z) = \sum_{l\in\mathbb{Z}}^{\infty} t^l\partial_t^r E_{i,j}z^{-l-1}. \tag{1.8}$$

The *vacuum module* $\mathcal{V}_\chi$ of $\widehat{gl}(n,\mathbb{A})$ is a module generated by a vector $|0\rangle$, called *vacuum*, such that $\kappa|_{\mathcal{V}_\chi} = \chi\,\mathrm{Id}_{\mathcal{V}_\chi}$,

$$E_{i,j}(r,z)|0\rangle = \sum_{m=0}^{\infty} t^{-m-1}\partial_t^r E_{i,j}|0\rangle z^m \tag{1.9}$$

for $i, j \in \overline{1,n}$, $r \in \mathbb{N}$, and any other $\widehat{gl}(n,\mathbb{A})$-module with the same property must be a quotient module of $\mathcal{V}_\chi$. Denote by $U(\cdot)$ the universal enveloping algebra of a Lie algebra. Suppose that $\mathcal{G}$ is one of the Lie algebras $\widehat{gl}(\vec{\ell},\mathbb{A})$, $\hat{o}(\vec{\ell},\mathbb{A})$ or $\widehat{sp}(\vec{\ell},\mathbb{A})$. The $\mathcal{G}$-module

$$\mathcal{V}_\chi(\mathcal{G}) = U(\mathcal{G})|0\rangle \tag{1.10}$$

is called the *vacuum module* of $\mathcal{G}$ and the corresponding representation is called the *vacuum representation* of $\mathcal{G}$. Our first main result in this paper is as follows:

Theorem 1.1. *The module $\mathcal{V}_\chi(\mathcal{G})$ is irreducible if $\chi \notin \mathbb{Z}$. When $\chi \in \mathbb{Z}$, the module $\mathcal{V}_\chi(\mathcal{G})$ has a unique maximal proper submodule $\bar{\mathcal{V}}_\chi(\mathcal{G})$, and the quotient $V_\chi(\mathcal{G}) = \mathcal{V}_\chi(\mathcal{G})/\bar{\mathcal{V}}_\chi(\mathcal{G})$ is an irreducible $\mathcal{G}$-module. When $n > 1$ and $\chi \in \mathbb{N}$, the submodule*

$$\bar{\mathcal{V}}_\chi(\mathcal{G}) = U(\mathcal{G})(t^{-1}\partial^{\ell_1}E_{n,1})^{\chi+1}|0\rangle \tag{1.11}$$

for $\mathcal{G} = \widehat{gl}(\vec{\ell},\mathbb{A})$, $\mathcal{G} = \hat{o}(\vec{\ell},\mathbb{A})$ with $\epsilon = 1$ and $\mathcal{G} = \widehat{sp}(\vec{\ell},\mathbb{A})$ with $\epsilon = 0$.

If $n > 3$, the submodule

$$\bar{\mathcal{V}}_\chi(\mathcal{G}) = U(\mathcal{G})(t^{-1}(\partial^{\ell_1}E_{n-1,1} - \partial_t^{\ell_2}E_{n,2}))^{\chi+1}|0\rangle \tag{1.12}$$

for $\mathcal{G} = \hat{o}(\vec{\ell},\mathbb{A})$ with $\epsilon = 0$ and $\mathcal{G} = \widehat{sp}(\vec{\ell},\mathbb{A})$ with $\epsilon = 1$.

The main difficulty of proving this theorem is to prove that the vacuum module $\mathcal{V}_\chi(\mathcal{G})$ is isomorphic to a certain highest-weight $\mathcal{G}$-module induced from a highest-weight module of the corresponding centrally-extended classical Lie algebra of infinite matrices with finite number of nonzero entries. This is done by calculating characters, which is not as trivial as it seems to be. Indeed, we have to use the results in our earlier work [34] in dealing with the cases $\mathcal{G} = \hat{o}(\vec{\ell},\mathbb{A}), \widehat{sp}(\vec{\ell},\mathbb{A})$.

Suppose that χ is a positive integer. Note that $(t^{-1}E_{n,1})^{\chi+1}|0\rangle$ is a singular vector generating the maximal proper submodules of the vacuum modules at level χ of the affine Kac–Moody algebras $\widehat{sl}(n,\mathbb{C})$ and $\widehat{sp}(n,\mathbb{C})$, respectively. Moreover, $(t^{-1}(E_{n-1,1}-E_{n,2}))^{\chi+1}|0\rangle$ is a singular vector generating the maximal proper submodules of the vacuum module at level χ of the affine Kac–Moody algebra $\hat{o}(n,\mathbb{C})$ when n is even. Our above results are exactly analogues in high order differential operators to those of the affine Kac–Moody algebras in [15].

On $\mathcal{V}_\chi$, there exists a unique vertex algebra structure whose structure map $Y(\cdot,z)$ satisfying $Y(|0\rangle,z)=\mathrm{Id}_{\mathcal{V}_\chi}$ and

$$Y(t^{-m-1}\partial^r E_{i,j},z)=\frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}E_{i,j}(r,z) \tag{1.13}$$

for $i,j\in\overline{1,n}$ and $r,m\in\mathbb{N}$. Let $\mathcal{G}$ be one of the Lie algebras $\widehat{gl}(\vec{\ell},\mathbb{A})$, $\hat{o}(\vec{\ell},\mathbb{A})$ or $\widehat{sp}(\vec{\ell},\mathbb{A})$. The family $(\mathcal{V}_\chi(\mathcal{G}),Y(\cdot,z))$ forms a vertex subalgebra. If $\chi\notin\mathbb{Z}$, the vertex algebra $(\mathcal{V}_\chi(\mathcal{G}),Y(\cdot,z))$ is simple. When $\chi\in\mathbb{Z}$, the quotient space $V_\chi(\mathcal{G})$ forms a simple vertex algebra. If $\ell_i\in\{0,1\}$ for $i\in\overline{1,n}$, the above vertex algebras have a Virasoro element, and thus they are vertex operator algebras.

Denote $\mathcal{Z}=\mathbb{Z}+1/2$. Let $\overline{gl}(\infty)$ be a vector space with a basis $\{\mathcal{E}_{l,k}\mid l,k\in\mathcal{Z}\}$ and multiplication:

$$\mathcal{E}_{l_1,l_2}\cdot\mathcal{E}_{k_1,k_2}=\delta_{l_2+k_1,0}\mathcal{E}_{l_1,k_2}\quad\text{for } l_1,l_2,k_1,k_2\in\mathcal{Z}. \tag{1.14}$$

Then $\overline{gl}(\infty)$ is isomorphic to the associative algebra of infinite matrices with finite number of nonzero entries. The reason for using $\delta_{l_2+k_1,0}$ instead of using δ_{l_2,k_1} in (1.14) is that we want our notation to coincide with that in affine Lie algebras and the free field theory. In this way, the conformal weight of $\mathcal{E}_{l_1,l_2}$ is $-(l_1+l_2)$ that is symmetric with respect to both subindices. Define the step function H on $\mathcal{Z}$ by

$$H(l)=\begin{cases}1 & \text{if } l>0,\\ 0 & \text{if } l<0\end{cases}\quad\text{for } l\in\mathcal{Z}. \tag{1.15}$$

Set $\tilde{gl}(\infty)=\overline{gl}(\infty)\oplus\mathbb{C}\kappa_0$, where κ_0 is the symbol for the base element. We have the following Lie bracket on $\tilde{gl}(\infty)$:

$$\begin{aligned}[\mathcal{E}_{l_1,l_2}+\mu_1\kappa_0,\mathcal{E}_{k_1,k_2}+\mu_2\kappa_0]&=\mathcal{E}_{l_1,l_2}\mathcal{E}_{k_1,k_2}-\mathcal{E}_{k_1,k_2}\mathcal{E}_{l_1,l_2}\\&\quad+\delta_{l_1+k_2,0}\delta_{l_2+k_1,0}[H(l_1)H(l_2)-H(k_1)H(k_2)]\kappa_0\end{aligned} \tag{1.16}$$

for $l_1,l_2,k_1,k_2\in\mathcal{Z}$ and $\mu_1,\mu_2\in\mathbb{C}$. Moreover, the subspace $\mathcal{T}=\sum_{l\in\mathcal{Z}}\mathbb{C}\mathcal{E}_{l,-l}+\mathbb{C}\kappa_0$ forms a toral Cartan subalgebra of $\tilde{gl}(\infty)$. In fact, $\overline{gl}(\infty)$ and $\tilde{gl}(\infty)$ have a set of Chevalley generators satisfying the same Serre's defining relations. Thus they have exactly the same representation theory.

Given $\mu\in\mathbb{C}$, we set $\langle\mu\rangle_0=1$ and $\langle\mu\rangle_m=\mu(\mu-1)\cdots(\mu-(m-1))$ for $0<m\in\mathbb{N}$. Denote by $\mathcal{T}^*$ the space of linear functions on $\mathcal{T}$. For $\lambda\in\mathcal{T}^*$, we define

$$\operatorname{supp}\lambda=\{l\in\mathcal{Z}\mid\lambda(\mathcal{E}_{l,-l})\neq 0\}. \tag{1.17}$$

Pick a weight λ such that $\lambda(\kappa_0)=\chi$ and $\operatorname{supp}\lambda$ is a finite set. Let $\mathcal{M}$ be the highest-weight irreducible $\tilde{gl}(\infty)$-module with weight λ, whose highest-weight vector is annihilated by the subalgebra $\operatorname{Span}\{\mathcal{E}_{l,k}\mid l,k\in\mathcal{Z};l+k>0\}$. Fix a constant $\iota\in\mathbb{C}$. We construct a module structure of the vertex algebra $\mathcal{V}_\chi$ whose structure map $Y^\iota_{\mathcal{M}}(\cdot,z)$ satisfies $Y^\iota_{\mathcal{M}}(|0\rangle,z)=\mathrm{Id}_{\mathcal{M}}$ and

$$Y_M(t^{-1}\partial^r E_{i,j},z)\equiv\sum_{l,k\in\mathbb{Z}}\langle\iota-k\rangle_r\mathcal{E}_{ln+i-1/2,kn-j+1/2}z^{-l-k-r-1}\pmod{\mathbb{C}\kappa_0} \tag{1.18}$$

if $\iota\notin\mathbb{Z}$, and

$$\begin{aligned}Y_M(t^{-1}\partial^r E_{i,j},z)\equiv\sum_{l,k=0}^{n}&[\langle -k-1\rangle_r\mathcal{E}_{(l-\iota)n+i-1/2,(k+\iota+1)n-j+1/2}z^{-l-k-\ell_i-r-2}\\&+\langle k+\ell_j\rangle_r\mathcal{E}_{(l-\iota)n+i-1/2,(\iota-k)n-j+/2}z^{-l+k+\ell_j-\ell_i-r-1}\\&+\langle k+\ell_j\rangle_r\mathcal{E}_{-(l+\iota+1)n+i-1/2,(\iota-k)n-j+1/2}z^{l+k+\ell_j-r-1}\\&+\langle -k-1\rangle_r\mathcal{E}_{-(l+\iota+1)n+i-1/2,(k+\iota+1)n-j+1/2}z^{l-k-r-1}]\pmod{\mathbb{C}\kappa_0}\end{aligned} \tag{1.19}$$

if $\iota\in\mathbb{Z}$.

Let m be a positive integer. Denote by $\mathcal{S}_m = \{\{3/2 - r, 5/2 - r, \ldots, (2m+1)/2 - r\} \mid r \in \overline{1, m+1}\}$ the set of intervals around 0 of length m in $\mathcal{Z}$. Define

$$\Gamma^m = \{\lambda \in \mathcal{T}^* \mid \lambda(\kappa_0) = -m, -s^{-1}|s|\lambda(\mathcal{E}_{s,-s}) \in \mathbb{N} \text{ for } s \in \mathcal{Z}; \operatorname{supp} \lambda \subset S \text{ for some } S \in \mathcal{S}_m\}. \tag{1.20}$$

Theorem 1.2. *The family $(\mathcal{M}, Y^{\iota}_{\mathcal{M}}(|_{\mathcal{V}_\chi(\mathcal{G})}, z))$ forms an irreducible module of the vertex algebra $(\mathcal{V}_\chi(\mathcal{G}), Y(\cdot, z))$ for $\mathcal{G} = \widehat{gl}(\vec{\ell}, \mathbb{A})$ and $\mathcal{G} = \hat{o}(\vec{\ell}, \mathbb{A}), \widehat{sp}(\vec{\ell}, \mathbb{A})$ if $\iota \notin \mathbb{Z}/2$.*

Suppose that χ is a positive integer and

$$\lambda(\mathcal{E}_{1/2,-1/2} - \mathcal{E}_{-1/2,1/2} + \kappa_0), \lambda(\mathcal{E}_{l+1,-l-1} - \mathcal{E}_{l,-l}) \in \mathbb{N} \tag{1.21}$$

for $-1/2 \neq l \in \mathcal{Z}$. The family $(\mathcal{M}, Y^{\iota}_{\mathcal{M}}(|_{\mathcal{V}_\chi(\mathcal{G})}, z))$ induces an irreducible module of the quotient simple vertex algebra $(V_\chi(\mathcal{G}), Y(\cdot, z))$ for $\mathcal{G} = \widehat{gl}(\vec{\ell}, \mathbb{A})$ and $\mathcal{G} = \hat{o}(\vec{\ell}, \mathbb{A}), \widehat{sp}(\vec{\ell}, \mathbb{A})$ if $\iota \notin \mathbb{Z}/2$.

Assume that $\chi = -m$ is a negative integer and $\lambda \in \Gamma^m$. The family $(\mathcal{M}, Y^{\iota}_{\mathcal{M}}(|_{\mathcal{V}_\chi(\mathcal{G})}, z))$ induces an irreducible module of the quotient simple vertex algebra $(V_{-m}(\mathcal{G}), Y(\cdot, z))$ for $\mathcal{G} = \widehat{gl}(\vec{\ell}, \mathbb{A})$ and $\mathcal{G} = \hat{o}(\vec{\ell}, \mathbb{A}), \widehat{sp}(\vec{\ell}, \mathbb{A})$ if $\iota \notin \mathbb{Z}/2$.

As in the work of Frenkel–Zhu [15] on vertex operator algebras associated with affine Kac–Moody algebras, it is difficult to determine when an irreducible highest-weight $\mathcal{G}$-module can be a module of the simple vertex algebra $(V_\chi(\mathcal{G}), Y(\cdot, z))$. We have used the tensor theory of vertex algebras associated with free fields and certain generalizations of their twisted modules. This is part of the reason why this paper is so long. Our method is completely different from that of Frenkel–Zhu, and indeed the case that χ is a negative integer for affine Kac–Moody algebras had not been handled in [15].

We remark that the module $\mathcal{M}$ is a unitary module if (1.21) is satisfied or $\lambda \in \Gamma^m$. When $\iota \in \mathbb{Z}/2$, we construct irreducible modules of the vertex algebras $(\mathcal{V}_\chi(\hat{o}(\vec{\ell}, \mathbb{A})), Y(\cdot, z))$ and $(\mathcal{V}_\chi(\widehat{sp}(\vec{\ell}, \mathbb{A})), Y(\cdot, z))$ through highest-weight modules of nonstandard centrally-extended other types of classical Lie algebras of infinite matrices with finite number of nonzero entries. Similar conclusions for their quotient simple vertex algebras hold. When χ is a positive integer, our results on modules are natural analogues of those for the simple vertex operator algebras associated with affine Lie algebras. When $n > 3$ and χ is a positive integer, the condition (1.21) can be relaxed to obtain certain nonunitary irreducible modules of the simple vertex algebras $V_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$, $V_\chi(\hat{o}(\vec{\ell}, \mathbb{A}))$ and $V_\chi(\widehat{sp}(\vec{\ell}, \mathbb{A}))$. In the case $\vec{\ell} = \vec{0}$, our theory coincides with the charged quadratic free bosonic field theory if $\chi = -1$, and with the charged quadratic free fermionic field theory if $\chi = 1$. If $\vec{\ell} = (1, 1, \ldots, 1)$, our irreducible modules of the vertex algebra $(V_{-1}(\hat{o}(\vec{\ell}, \mathbb{A})), Y(\cdot, z))$ include those studied by Dong and Nagatomo [9,10].

The paper is organized as follows. In Section 2, we present the framework of constructing representations of certain Lie subalgebras of the centrally-extended Lie algebra of the tensor product of any associative algebra with the algebra of differential operators on the circle. Section 3 is devoted to the constructions of irreducible modules of the Lie algebras $\widehat{gl}(\vec{\ell}, \mathbb{A})$, $\hat{o}(\vec{\ell}, \mathbb{A})$ or $\widehat{sp}(\vec{\ell}, \mathbb{A})$ from weighted irreducible modules of the centrally-extended general linear Lie algebra of infinite matrices with finite number of nonzero entries. In Section 4, we give detailed constructions of irreducible modules with the parameter $\iota \in \mathbb{Z} + 1/2$ of the Lie algebras $\hat{o}(\vec{\ell}, \mathbb{A})$ and $\widehat{sp}(\vec{\ell}, \mathbb{A})$ from weighted irreducible modules of certain central extensions of the Lie algebras of infinite skew matrices with finite number of nonzero entries. The cases with the parameter $\iota \in \mathbb{Z}$ are handled in Section 5. In Section 6, we study the vacuum representation of the Lie algebra $\widehat{gl}(\vec{\ell}, \mathbb{A})$, its vertex algebra structure and vertex algebra irreducible representations. We deal with the cases for the Lie algebras $\hat{o}(\vec{\ell}, \mathbb{A})$ and $\widehat{sp}(\vec{\ell}, \mathbb{A})$ in Section 7.

2. General frame

In this section, we construct irreducible representations of certain Lie subalgebras of the centrally-extended Lie algebra of the tensor product of any associative algebra with the algebra of differential operators on the circle, from representations of the certain Lie subalgebras of a centrally-extended Lie algebra of the tensor product of the associative algebra with the algebra of infinite matrices with finite number of nonzero entries. When the associative algebra is a finite matrix algebra, the latter Lie algebras are exactly the centrally-extended classical Lie algebra of infinite matrices with finite number of nonzero entries.

Recall

$$\partial_t = \frac{\mathrm{d}}{\mathrm{d}t} \tag{2.1}$$

and the algebra of differential operators on the circle:

$$\mathbb{A} = \sum_{i=0}^{\infty} \mathbb{C}[t, t^{-1}]\partial_t^i. \tag{2.2}$$

Note that

$$f(t)\partial_t^i \cdot g(t)\partial_t^j = \sum_{r=0}^{i} \binom{i}{r} f(t)\frac{\mathrm{d}^r g}{\mathrm{d}t^r}(t)\partial_t^{i+j-r} \quad \text{for } f(t), g(t) \in \mathbb{C}[t, t^{-1}],\ i, j \in \mathbb{N}. \tag{2.3}$$

Let $\mathcal{A}$ be an associative algebra with an identity element $1_{\mathcal{A}}$ and a linear map $\mathrm{tr} : \mathcal{A} \to \mathbb{C}$ such that

$$\mathrm{tr}\, 1_{\mathcal{A}} \neq 0, \qquad \mathrm{tr}\, ab = \mathrm{tr}\, ba \quad \text{for } a, b \in \mathcal{A}. \tag{2.4}$$

Such a map "tr" is called a *trace map*. Set

$$\hat{\mathcal{A}} = \mathcal{A} \otimes_{\mathbb{C}} \mathbb{A} \oplus \mathbb{C}\kappa, \tag{2.5}$$

where κ is a base element. According to [27] (also cf. [28]), we have the following Lie bracket on $\hat{\mathcal{A}}$:

$$\begin{aligned}[][a \otimes t^{i_1}\partial_t^{j_1} + \mu_1\kappa, b \otimes t^{i_2}\partial_t^{j_2} + \mu_1\kappa] &= ab \otimes t^{i_1}\partial_t^{j_1} \cdot t^{i_2}\partial_t^{j_2} - ba \otimes t^{i_2}\partial_t^{j_2} \cdot t^{i_1}\partial_t^{j_1} \\ &\quad + (-1)^{j_1}\delta_{i_1+i_2, j_1+j_2} j_1! j_2! \binom{i_1}{j_1 + j_2 + 1} (\mathrm{tr}\, ab)\kappa \end{aligned} \tag{2.6}$$

for $a, b \in \mathcal{A}$, $i_1, i_2 \in \mathbb{Z}$, $j_1, j_2 \in \mathbb{N}$ and $\mu_1, \mu_2 \in \mathbb{C}$.

For two vector spaces V_1 and V_2, we denote by $LM(V_1, V_2)$ the space of linear maps from V_1 to V_2. We also use the following operator for taking residue:

$$\mathrm{Res}_z(z^n) = \delta_{n,-1} \quad \text{for } n \in \mathbb{Z}. \tag{2.7}$$

Furthermore, all the binomials are assumed to be expanded in the nonnegative powers of the second variable.

A *conformal algebra* R is a $\mathbb{C}[\partial]$-module with a linear map $Y^+(\cdot, z) : R \to LM(R, R[z^{-1}]z^{-1})$ satisfying:

$$Y^+(\partial u, z) = \frac{\mathrm{d}Y^+(u, z)}{\mathrm{d}z} \quad \text{for } u \in R; \tag{2.8}$$

$$Y^+(u, z)v = \mathrm{Res}_x \frac{e^{x\partial} Y^+(v, -x)u}{z - x}, \tag{2.9}$$

$$Y^+(u, z_1)Y^+(v, z_2) - Y^+(v, z_2)Y^+(u, z_1) = \mathrm{Res}_x \frac{Y^+(Y^+(u, z_1 - x)v, x)}{z_2 - x} \tag{2.10}$$

for $u, v \in R$. We denote by $(R, \partial, Y^+(\cdot, z))$ a conformal algebra.

Define

$$\hat{R}(\mathcal{A}) = \mathcal{A}[\varsigma_1, \varsigma_2] \oplus \mathbb{C}\mathbf{1}, \tag{2.11}$$

where ς_1, ς_2 are indeterminates and $\mathbf{1}$ is a base element. For convenience, we denote

$$a[m_1, m_2] = a\varsigma_1^{m_1}\varsigma_2^{m_2} \quad \text{for } a \in \mathcal{A}, m_1, m_2 \in \mathbb{N}. \tag{2.12}$$

We define the action of $\mathbb{C}[\partial]$ on $\hat{R}(\mathcal{A})$ by:

$$\partial(\mathbf{1}) = 0, \quad \partial(a[m_1, m_2]) = (m_1 + 1)a[m_1 + 1, m_2] + (m_2 + 1)a[m_1, m_2 + 1], \tag{2.13}$$

and a linear map $Y^+(\cdot, z) : \hat{R}(\mathcal{A}) \to LM(\hat{R}(\mathcal{A}), \hat{R}(\mathcal{A})[z^{-1}]z^{-1})$ by

$$Y^+(w, z)\mathbf{1} = Y^+(\mathbf{1}, z)w = 0 \quad \text{for } w \in \hat{R}(\mathcal{A}) \tag{2.14}$$

and

$$\begin{aligned} Y^+(a[m_1, m_2], z)b[n_1, n_2] = & \binom{-n_1-1}{m_2} \sum_{p=0}^{m_1+m_2+n_1} \binom{p}{m_1} ab[p, n_2] z^{p-m_1-m_2-n_1-1} \\ & - \binom{-n_2-1}{m_1} \sum_{q=0}^{m_1+m_2+n_2} \binom{q}{m_2} ba[n_1, q] z^{q-m_1-m_2-n_2-1} \\ & + \binom{-n_1-1}{m_2}\binom{-n_2-1}{m_1} (\mathrm{tr}\, ab)\mathbf{1} z^{-m_1-m_2-n_1-n_2-2} \end{aligned} \tag{2.15}$$

for $a, b \in \mathcal{A}$ and $m_1, m_2, n_1, n_2 \in \mathbb{N}$. Then $(\hat{R}(\mathcal{A}), \partial, Y^+(\cdot, z))$ forms a conformal algebra (cf. Section 7.3 in [33]).

Let

$$\hat{\mathcal{A}}[[z^{-1}, z]] = \left\{ \sum_{m\in\mathbb{Z}} u_m z^m \mid u_m \in \hat{\mathcal{A}} \right\} \tag{2.16}$$

be the space of formal power series with coefficients in $\hat{\mathcal{A}}$. Define a linear map $Y(\cdot, z) : \hat{R}(\mathcal{A}) \to \hat{\mathcal{A}}[[z^{-1}, z]]$ by

$$Y(\mathbf{1}, z) = \kappa, \quad Y(a[m_1, m_2], z) = \frac{1}{m_1! m_2!} \sum_{i\in\mathbb{Z}} a \otimes (-\partial_t)^{m_1} t^i \partial_t^{m_2} z^{-i-1} \tag{2.17}$$

for $a \in \mathcal{A}$ and $m_1, m_2 \in \mathbb{N}$. By Lemma 3.1 in [28], we have:

Lemma 2.1. *The Lie bracket (2.6) on $\hat{\mathcal{A}}$ is equivalent to:*

$$[Y(u, z_1), Y(v, z_2)] = Res_{z_0} z_1^{-1} \delta\left(\frac{z_2 + z_0}{z_1}\right) Y(Y^+(u, z_0)v, z_2) \tag{2.18}$$

for $u, v \in \hat{R}(\mathcal{A})$.

Recall

$$\mathcal{Z} = \mathbb{Z} + \frac{1}{2} \tag{2.19}$$

and the step function H on $\mathcal{Z}$:

$$H(l) = \begin{cases} 1 & \text{if } l > 0, \\ 0 & \text{if } l < 0 \end{cases} \quad \text{for } l \in \mathcal{Z}. \tag{2.20}$$

Set

$$\check{\mathcal{A}} = \sum_{i,j\in\mathcal{Z}} \mathcal{A} t_1^i t_2^j \oplus \mathbb{C}\kappa_0, \tag{2.21}$$

where t_1, t_2 are indeterminates and κ_0 is a base element. For convenience, we denote

$$a(i, j) = a t_1^i t_2^j \quad \text{for } a \in \mathcal{A}, i, j \in \mathcal{Z}. \tag{2.22}$$

According to Section 7.3 in [33], we have the following Lie bracket on $\check{\mathcal{A}}$:

$$\begin{aligned} [a(l_1, l_2) + \mu_1\kappa_0, b(k_1, k_2) + \mu_2\kappa_0] = & \delta_{l_2+k_1,0}\, ab(l_1, k_2) - \delta_{l_1+k_2,0}\, ba(k_1, l_2) \\ & + \delta_{l_1+k_2,0}\delta_{l_2+k_1,0}[H(l_1)H(l_2) - H(k_1)H(k_2)](\mathrm{tr}\, ab)\kappa_0 \end{aligned} \tag{2.23}$$

for $a, b \in \mathcal{A}, l_1, l_2, k_1, k_2 \in \mathcal{Z}$ and $\mu_1, \mu_2 \in \mathbb{C}$. The algebra $\check{\mathcal{A}}$ is isomorphic to the corresponding central extension of the commutator Lie algebra of $\mathcal{A} \otimes \overline{gl}(\infty)$, where we have the correspondence $t_1^{l_1} t_2^{l_2} \leftrightarrow \mathcal{E}_{l_1,l_2}$ (cf. (1.14) and the above).

Set

$$\check{\mathcal{A}}^m = \mathrm{Span}\{a(i,j) \mid a \in \mathcal{A}, i, j \in \mathcal{Z}; m < |i|, |j|; ij < 0\} \quad \text{for } 0 < m \in \mathbb{N}. \tag{2.24}$$

It can be verified that $\check{\mathcal{A}}^m$ is a Lie subalgebra of $\check{\mathcal{A}}$. Suppose that $\mathcal{M}$ is an $\check{\mathcal{A}}$-module

$$\text{generated by a subspace } \mathcal{M}_0 \text{ such that } \check{\mathcal{A}}^m(\mathcal{M}_0) = \{0\} \text{ for some } m \in \mathbb{N}. \tag{2.25}$$

Fix a constant $\iota \in \mathbb{C}$. Define

$$\begin{aligned}\sum_{r_1,r_2=0}^{\infty} \Im_{r_1,r_2} x^{r_1} y^{r_2} z^{-r_1-r_2-1} &= \frac{1}{x-y}\left(\left(\frac{z+y}{z+x}\right)^{\iota} - 1\right)\\ &= -\left[\sum_{r=2}^{\infty}\binom{\iota}{r}\frac{x^{r-1}+x^{r-2}y+\cdots y^{r-1}}{z^r} + \frac{\iota}{z}\right]\left[\sum_{s=0}^{\infty}\binom{-\iota}{s}\left(\frac{x}{z}\right)^s\right].\end{aligned} \tag{2.26}$$

Motivated by the construction of the twisted modules of spinor vertex operator algebras in [32], we define a linear map $Y_{\mathcal{M}}^{\iota}(\cdot, z) : \hat{R}(\mathcal{A}) \to LM(\mathcal{M}, \mathcal{M}[[z^{-1}, z]])$ by

$$Y_{\mathcal{M}}^{\iota}(\mathbf{1}, z) = \kappa_0, \tag{2.27}$$

$$\begin{aligned}Y_{\mathcal{M}}^{\iota}(a[r_1,r_2], z) = &\sum_{i,j\in\mathcal{Z}}\binom{-i-\iota-1/2}{r_1}\binom{-j+\iota-1/2}{r_1} a(i,j) z^{-i-j-r_1-r_2-1}\\ &+ \Im_{r_1,r_2}(\mathrm{tr}\, a)\kappa_0 z^{-r_1-r_2-1}\end{aligned} \tag{2.28}$$

for $a \in \mathcal{A}$ and $r_1, r_2 \in \mathbb{N}$. The above expression makes sense because of (2.25). Denote

$$\langle\mu\rangle_0 = 1, \qquad \langle\mu\rangle_m = \mu(\mu-1)\cdots(\mu-(m-1)) \quad \text{for } \mu \in \mathbb{C}, 0 < m \in \mathbb{N}. \tag{2.29}$$

Theorem 2.2. *On the $\check{\mathcal{A}}$-module $\mathcal{M}$, we have*

$$[Y_{\mathcal{M}}^{\iota}(u, z_1), Y_{\mathcal{M}}^{\iota}(v, z_2)] = Res_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right) Y_{\mathcal{M}}^{\iota}(Y^+(u, z_0)v, z_2) \tag{2.30}$$

for $u, v \in \hat{R}(\mathcal{A})$. In particular, Lemma 2.1 and (2.30) imply that $\mathcal{M}$ provides the following representation $\sigma_{\mathcal{M}}^{\iota}$ of the Lie algebra $\hat{\mathcal{A}}$:

$$\sigma_{\mathcal{M}}^{\iota}(\kappa) = \kappa_0, \tag{2.31}$$

$$\sigma_{\mathcal{M}}^{\iota}(a \otimes t^m \partial_t^r) = \sum_{l\in\mathcal{Z}}\langle -l+\iota-1/2\rangle_r a(m-r-l, l) + r!\delta_{r,m}\Im_{0,r}(tr\, a)\kappa_0 \tag{2.32}$$

for $m \in \mathbb{Z}, r \in \mathbb{N}$ and $a \in \mathcal{A}$.

Proof. We shall use generating functions to prove the above theorem. Set

$$c[x, y] = \sum_{r_1,r_2=0}^{\infty} c[r_1, r_2] x^{r_1} y^{r_2} \quad \text{for } c \in \mathcal{A}. \tag{2.33}$$

Fix $a, b \in \mathcal{A}$ and let

$$u = a[x_1, x_2], \qquad v = b[y_1, y_2]. \tag{2.34}$$

Then

$$Y^+(u,z_0)v = \frac{1}{z_0+x_2-y_1}ab[z_0+x_1,y_2] - \frac{1}{z_0+x_1-y_2}ba[y_1,z_0+x_2] + \frac{1}{(z_0+x_2-y_1)(z_0+x_1-y_2)}(\mathrm{tr}\,ab)\mathbf{1} \tag{2.35}$$

by (2.15).

On the other hand, we define

$$c(z_1,z_2) = \sum_{i,j\in\mathcal{Z}} c(i,j) z_1^{-i-\iota-1/2} z_2^{-j+\iota-1/2} \quad \text{for } c\in\mathcal{A}. \tag{2.36}$$

Then

$$Y^\iota_{\mathcal{M}}(u,z_1) = a(z_1+x_1,z_1+x_2) + \frac{1}{x_1-x_2}\left(\left(\frac{z_1+x_2}{z_1+x_1}\right)^\iota - 1\right)(\mathrm{tr}\,a)\kappa_0, \tag{2.37}$$

$$Y^\iota_{\mathcal{M}}(v,z_1) = b(z_2+y_1,z_2+y_2) + \frac{1}{y_1-y_2}\left(\left(\frac{z_2+y_2}{z_2+y_1}\right)^\iota - 1\right)(\mathrm{tr}\,a)\kappa_0. \tag{2.38}$$

Hence

$$\begin{aligned}
[Y^\iota_{\mathcal{M}}(u,z_1), Y^\iota_{\mathcal{M}}(v,z_1)] &= [a(z_1+x_1,z_1+x_2), b(z_2+y_1,z_2+y_2)] \\
&= (z_2+y_1)^{-1}\left(\frac{z_1+x_2}{z_2+y_1}\right)^\iota \delta\left(\frac{z_1+x_2}{z_2+y_1}\right) ab(z_1+x_1,z_2+y_2) \\
&\quad - (z_2+y_2)^{-1}\left(\frac{z_2+y_2}{z_1+x_1}\right)^\iota \delta\left(\frac{z_1+x_1}{z_2+y_2}\right) ba(z_2+y_1,z_1+x_2) \\
&\quad + \left(\frac{(z_1+x_2)(z_2+y_2)}{(z_1+x_1)(z_2+y_1)}\right)^\iota \left[\frac{1}{(z_1+x_1-z_2-y_2)(z_1+x_2-z_2-y_1)}\right. \\
&\quad \left. - \frac{1}{(z_2+y_1-z_1-x_2)(z_2+y_2-z_1-x_1)}\right](\mathrm{tr}\,ab)\kappa_0.
\end{aligned} \tag{2.39}$$

Note that

$$\begin{aligned}
&\left(\frac{(z_1+x_2)(z_2+y_2)}{(z_1+x_1)(z_2+y_1)}\right)^\iota \left[\frac{1}{(z_1+x_1-z_2-y_2)(z_1+x_2-z_2-y_1)}\right. \\
&\quad \left. - \frac{1}{(z_2+y_1-z_1-x_2)(z_2+y_2-z_1-x_1)}\right] \\
&= \frac{1}{x_1+y_1-x_2-y_2}\left(\frac{(z_1+x_2)(z_2+y_2)}{(z_1+x_1)(z_2+y_1)}\right)^\iota \\
&\quad \times \left[\frac{1}{z_1+x_2-z_2-y_1} - \frac{1}{z_1+x_1-z_2-y_2} + \frac{1}{z_2+y_1-z_1-x_2} - \frac{1}{z_2+y_2-z_1-x_1}\right] \\
&= \frac{z_1^{-1}}{x_1+y_1-x_2-y_2}\left(\frac{(z_1+x_2)(z_2+y_2)}{(z_1+x_1)(z_2+y_1)}\right)^\iota \left[\delta\left(\frac{z_2+y_1-x_2}{z_1}\right) - \delta\left(\frac{z_2+y_2-x_1}{z_1}\right)\right] \\
&= \frac{z_1^{-1}}{x_1+y_1-x_2-y_2}\left[\left(\frac{z_2+y_2}{z_2+x_1+y_1-x_2}\right)^\iota \delta\left(\frac{z_2+y_1-x_2}{z_1}\right)\right. \\
&\quad \left. - \left(\frac{z_2+x_2+y_2-x_1}{z_2+y_1}\right)^\iota \delta\left(\frac{z_2+y_2-x_1}{z_1}\right)\right].
\end{aligned} \tag{2.40}$$

Observe that

$$\begin{aligned}
(z_2+y_1)^{-1}\delta\left(\frac{z_1+x_2}{z_2+y_1}\right) &= \frac{1}{z_1+x_2-z_2-y_1} + \frac{1}{z_2+y_1-z_1-x_2} \\
&= z_1^{-1}\delta\left(\frac{z_2+y_1-x_2}{z_1}\right).
\end{aligned} \tag{2.41}$$

Hence

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\frac{1}{z_0+x_2-y_1}ab(z_2+z_0+x_1,z_2+y_2)\\
&\quad=\mathrm{Res}_{z_0}\frac{(z_2+y_1)^{-\iota}}{z_0+x_2-y_1}z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)[(z_2+y_1)^{\iota}ab(z_2+z_0+x_1,z_2+y_2)]\\
&\quad=\mathrm{Res}_{z_0}\frac{(z_2+y_1)^{-\iota}}{z_0+x_2-y_1}z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)[(z_1-z_0+y_1)^{\iota}ab(z_1+x_1,z_2+y_2)]\\
&\quad=\left(\frac{z_1+x_2}{z_2+y_1}\right)^{\iota}z_1^{-1}\delta\left(\frac{z_2+y_1-x_2}{z_1}\right)ab(z_1+x_1,z_2+y_2)\\
&\quad=(z_2+y_1)^{-1}\left(\frac{z_1+x_2}{z_2+y_1}\right)^{\iota}\delta\left(\frac{z_1+x_2}{z_2+y_1}\right)ab(z_1+x_1,z_2+y_2).
\end{aligned}\tag{2.42}$$

Similarly, we have

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\frac{1}{z_0+x_1-y_2}ba(z_2+y_1,z_2+z_0+x_2)\\
&\quad=(z_2+y_2)^{-1}\left(\frac{z_2+y_2}{z_1+x_1}\right)^{\iota}\delta\left(\frac{z_1+x_1}{z_2+y_2}\right)ba(z_2+y_1,z_1+x_2).
\end{aligned}\tag{2.43}$$

Moreover,

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\frac{1}{(z_0+x_2-y_1)(z_0+x_1-y_2)}\left[\left(\frac{z_2+y_2}{z_2+z_0+x_1}\right)^{\iota}+\left(\frac{z_2+z_0+x_2}{z_2+y_1}\right)^{\iota}-1\right]\\
&\quad=\frac{z_1^{-1}}{x_1+y_1-x_2-y_2}\left\{\mathrm{Res}_{z_0}\frac{1}{z_0+x_2-y_1}\delta\left(\frac{z_2+z_0}{z_1}\right)\left[\left(\frac{z_2+y_2}{z_2+z_0+x_1}\right)^{\iota}+\left(\frac{z_2+z_0+x_2}{z_2+y_1}\right)^{\iota}-1\right]\right.\\
&\qquad\left.-\mathrm{Res}_{z_0}\frac{1}{z_0+x_1-y_2}\delta\left(\frac{z_2+z_0}{z_1}\right)\left[\left(\frac{z_2+y_2}{z_2+z_0+x_1}\right)^{\iota}+\left(\frac{z_2+z_0+x_2}{z_2+y_1}\right)^{\iota}-1\right]\right\}\\
&=\frac{z_1^{-1}}{x_1+y_1-x_2-y_2}\left[\delta\left(\frac{z_2+y_1-x_2}{z_1}\right)\left(\frac{z_2+y_2}{z_2+x_1+y_1-x_2}\right)^{\iota}\right.\\
&\qquad\left.-\delta\left(\frac{z_2+y_2-x_1}{z_1}\right)\left(\frac{z_2+x_2+y_2-x_1}{z_2+y_1}\right)^{\iota}\right].
\end{aligned}\tag{2.44}$$

By (2.35), (2.39), (2.40) and (2.42)–(2.44), we have

$$\begin{aligned}
&\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)Y^{\iota}_{\mathcal{M}}(Y^{+}(u,z_0)v,z_2)\\
&\quad=\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\left[\frac{1}{z_0+x_2-y_1}Y^{\iota}_{\mathcal{M}}(ab[z_0+x_1,y_2],z_2)\right.\\
&\qquad\left.-\frac{1}{z_0+x_1-y_2}Y^{\iota}_{\mathcal{M}}(ba[y_1,z_0+x_2],z_2)+\frac{1}{(z_0+x_2-y_1)(z_0+x_1-y_2)}(\mathrm{tr}\,ab)\kappa_0\right]\\
&\quad=\mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\left\{\frac{1}{z_0+x_2-y_1}\left[ab(z_2+z_0+x_1,z_2+y_2)\right.\right.\\
&\qquad\left.+\frac{1}{z_0+x_1-y_2}\left[\left(\frac{z_2+y_2}{z_2+z_0+x_1}\right)^{\iota}-1\right](\mathrm{tr}\,ab)\kappa_0\right]\\
&\qquad-\frac{1}{z_0+x_1-y_2}\left[ba(z_2+y_1,z_2+z_0+x_2)-\frac{1}{z_0+x_2-y_1}\right.\\
&\qquad\left.\left.\times\left[\left(\frac{z_2+z_0+x_2}{z_2+y_1}\right)^{\iota}-1\right](\mathrm{tr}\,ab)\kappa_0\right]+\frac{1}{(z_0+x_2-y_1)(z_0+x_1-y_2)}(\mathrm{tr}\,ab)\kappa_0\right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\frac{1}{z_0+x_2-y_1} ab(z_2+z_0+x_1, z_2+y_2) \\
&\quad - \mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\frac{1}{z_0+x_1-y_2} ba(z_2+y_1, z_2+z_0+x_2) \\
&\quad + \mathrm{Res}_{z_0} z_1^{-1}\delta\left(\frac{z_2+z_0}{z_1}\right)\frac{1}{(z_0+x_2-y_1)(z_0+x_1-y_2)} \\
&\quad \times\left[\left(\frac{z_2+y_2}{z_2+z_0+x_1}\right)^{\iota}+\left(\frac{z_2+z_0+x_2}{z_2+y_1}\right)^{\iota}-1\right] \\
&= [Y^{\iota}_{\mathcal{M}}(u,z_1), Y^{\iota}_{\mathcal{M}}(v,z_2)],
\end{aligned} \tag{2.45}$$

that is, (2.30) holds. Moreover, (2.32) follows from (2.17), Lemma 2.1 and (2.30). □

Suppose that the associative algebra $\mathcal{A}$ has n left ideals $\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n\}$ such that

$$\mathcal{A} = \bigoplus_{i=1}^{n} \mathcal{B}_i. \tag{2.46}$$

Take

$$\vec{\ell} = (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{N}^n \tag{2.47}$$

and set

$$\hat{\mathcal{A}}_{\vec{\ell}} = \sum_{i=1}^{n} \mathcal{B}_i \otimes \mathbb{A}\partial_t^{\ell_i} + \mathbb{C}\kappa. \tag{2.48}$$

Then $\hat{\mathcal{A}}_{\vec{\ell}}$ forms a Lie subalgebra of $\hat{\mathcal{A}}$ (cf. (2.5) and (2.6)). Let $\mathcal{M}$ be an $\check{\mathcal{A}}$-module satisfying (2.25). Consider the restricted representation $\sigma^{\iota}_{\mathcal{M}}|_{\hat{\mathcal{A}}_{\vec{\ell}}}$ of $\hat{\mathcal{A}}_{\vec{\ell}}$. However, when $\iota \in \mathbb{Z}$ and $\vec{\ell} \neq \vec{0}$, some of the coefficients $\langle -l+\iota-1/2\rangle_r$ in (2.32) would be zero and the related elements $a(m-r-l, l)$ would yield obvious proper submodules. We want to exclude these elements $a(m-r-l, l)$ and the obvious proper submodules caused by them in order to get irreducible representations later on. Set

$$a(r,z) = \sum_{m\in\mathbb{Z}} a\otimes t^m\partial_t^r z^{-m-1} \qquad \text{for } a \in \mathcal{A},\, r \in \mathbb{N}. \tag{2.49}$$

Suppose that $\iota \in \mathbb{Z}$ and each

$$\mathcal{B}_j = \bigoplus_{i=1}^{n} \mathcal{B}_{i,j} \tag{2.50}$$

has a subspace decomposition such that

$$\mathcal{B}_{i,j}\cdot\mathcal{B}_{l,k} = \{0\} \quad \text{if } j \neq l. \tag{2.51}$$

By deleting those terms $\langle -l+\iota-1/2\rangle_r a(m-r-l, l)$ with $\langle -l+\iota-1/2\rangle_r = 0$ in (2.32) and partially shifting indices, we obtain:

Theorem 2.3. *On the $\check{\mathcal{A}}$-module $\mathcal{M}$, we have the following representation $\sigma_{\vec{\ell},\mathcal{M}}$ of $\hat{\mathcal{A}}_{\vec{\ell}}$: $\sigma_{\vec{\ell},\mathcal{M}}(\kappa) = \kappa_0$ and*

$$\begin{aligned}
\sigma_{\vec{\ell},\mathcal{M}}(a(r,z)) = \sum_{0<l_1,l_2\in\mathcal{Z}} &[\langle -l_2-1/2\rangle_r a(l_1-\iota, l_2+\iota) z^{-l_1-l_2-\ell_i-r-1} + \langle l_2+\ell_j-1/2\rangle_r a(l_1-\iota, \iota-l_2) \\
&\times z^{-l_1+l_2+\ell_j-\ell_i-r-1} + \langle -l_2-1/2\rangle_r a(-l_1-\iota, l_2+\iota) z^{l_1-l_2-r-1} \\
&+ \langle l_2+\ell_j-1/2\rangle_r a(-l_1-\iota, \iota-l_2) z^{l_1+l_2+\ell_j-r-1}] + r!\Im_{0,r}(tr\, a)\kappa_0 z^{-r-1}
\end{aligned} \tag{2.52}$$

for $a \in \mathcal{B}_{i,j}$ and $r \in \mathbb{N}$.

Suppose that τ is a linear transformation on $\mathcal{A}$ such that

$$\tau^2 = \mathrm{Id}_{\mathcal{A}}, \qquad \tau(1_{\mathcal{A}}) = 1_{\mathcal{A}}, \qquad \tau(ab) = \tau(b)\tau(a) \quad \text{for } a, b \in \mathcal{A}, \tag{2.53}$$

$$\mathrm{tr}\,\tau = \mathrm{tr}\,, \qquad \tau(\mathcal{B}_{i,j}) \subset \mathcal{B}_{\pi(j),\pi(i)} \tag{2.54}$$

for $i, j \in \overline{1, n}$, where

$$\pi \quad \text{is a permutation on } \{1, 2, \ldots, n\}. \tag{2.55}$$

Let

$$\epsilon \in \{0, 1\} \tag{2.56}$$

and take $\vec{\ell} \in \mathbb{N}^n$ such that

$$\{\ell_1, \ell_2, \ldots, \ell_n\} \subset 2\mathbb{Z} + \epsilon \tag{2.57}$$

and

$$\ell_i = \ell_{\pi(i)} \quad \text{for } i \in \overline{1, n}. \tag{2.58}$$

Set

$$\hat{\mathcal{A}}^{\tau}_{\vec{\ell}} = \mathrm{Span}\{a \otimes t^m \partial_t^{r+\ell_j} - (-1)^{\epsilon}\tau(a) \otimes (-\partial_t)^r t^m \partial_t^{\ell_i} \mid i, j \in \overline{1, n}; a \in \mathcal{B}_{i,j}; r \in \mathbb{N}, m \in \mathbb{Z}\} + \mathbb{C}\kappa. \tag{2.59}$$

It can be verified that $\hat{\mathcal{A}}^{\tau}_{\vec{\ell}}$ forms a Lie subalgebra of $\hat{\mathcal{A}}$ (cf. (2.6), [28]). For any $\check{\mathcal{A}}$-module $\mathcal{M}$ satisfying (2.25), we have the restricted representation $\sigma^{\iota}_{\mathcal{M}}|_{\hat{\mathcal{A}}^{\tau}_{\vec{\ell}}}$. For convenience, we set

$$a^{\tau}_{\vec{\ell}}(r, z) = \sum_{m\in\mathbb{Z}} (a \otimes t^m \partial_t^{r+\ell_j} - (-1)^{\epsilon}\tau(a) \otimes (-\partial_t)^r t^m \partial_t^{\ell_i}) z^{-m-1}. \tag{2.60}$$

Then for $a \in \mathcal{B}_{i,j}$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}
\sigma^{\iota}_{\mathcal{M}}(a^{\tau}_{\vec{\ell}}(r, z)) &= Y^{\iota}_{\mathcal{M}}((r + \ell_j)!a[0, r + \ell_j] - (-1)^{\epsilon} r!\ell_i!\tau(a)[r, \ell_i], z) \\
&= \sum_{l_1,l_2\in\mathcal{Z}} \langle -l_2 + \iota - 1/2\rangle_{r+\ell_j} a(l_1, l_2) z^{-l_1-l_2-r-\ell_j-1} + (r + \ell_j)!\Im_{0,r+\ell_j}(\mathrm{tr}\, a)\kappa_0 z^{-r-\ell_j-1} \\
&\quad - (-1)^{\epsilon} \sum_{l_1,l_2\in\mathcal{Z}} \langle -l_1 - \iota - 1/2\rangle_r \langle -l_2 + \iota - 1/2\rangle_{\ell_i} \tau(a)(l_1, l_2) z^{-l_1-l_2-r-\ell_i-1} \\
&\quad - (-1)^{\epsilon} r!\ell_i!\Im_{r,\ell_i}(\mathrm{tr}\, a)\kappa_0 z^{-r-\ell_i-1} \\
&= \sum_{m\in\mathbb{Z}} \sum_{l\in\mathcal{Z}} [\langle l + r + \ell_j + \iota - 1/2\rangle_{r+\ell_j} a(m + l, -l - r - \ell_j) - (-1)^{\epsilon}\langle l + r - \iota - 1/2\rangle_r \\
&\quad \times \langle -m - l + \ell_i + \iota - 1/2\rangle_{\ell_i} \tau(a)(-l - r, m + l - \ell_i)] z^{-m-1} \\
&\quad + (r + \ell_j)!\Im_{0,r+\ell_j}(\mathrm{tr}\, a)\kappa_0 z^{-r-\ell_j-1} - (-1)^{\epsilon} r!\ell_i!\Im_{r,\ell_i}(\mathrm{tr}\, a)\kappa_0 z^{-r-\ell_i-1}
\end{aligned} \tag{2.61}$$

for $a \in \mathcal{B}_{i,j}$ and $r \in \mathbb{N}$ by (2.17) and (2.18). Thus

$$\begin{aligned}
&\sigma^{\iota}_{\mathcal{M}}(a \otimes t^m \partial_t^{r+\ell_j} - (-1)^{\epsilon}\tau(a) \otimes (-\partial_t)^r t^m \partial_t^{\ell_i}) \\
&\quad = \sum_{l\in\mathcal{Z}} [\langle l + r + \ell_j + \iota - 1/2\rangle_{r+\ell_j} a(m + l, -l - r - \ell_j) \\
&\quad - (-1)^{\epsilon}\langle l + r - \iota - 1/2\rangle_r \langle -m - l + \ell_i + \iota - 1/2\rangle_{\ell_i} \tau(a)(-r - l, m + l - \ell_i)] \\
&\quad + [(r + \ell_j)!\delta_{m,r+\ell_j}\Im_{0,r+\ell_j} - (-1)^{\epsilon} r!\ell_i!\delta_{m,r+\ell_i}\Im_{r,\ell_i}](\mathrm{tr}\, a)\kappa_0.
\end{aligned} \tag{2.62}$$

Suppose $\iota\in\mathbb{Z}/2$. Then the restricted representation of $\hat{\mathcal{A}}_{\vec{\ell}}^{\tau}$ is in general reducible. In (2.62),

$$\begin{aligned}&\sum_{l\in\mathcal{Z}}[\langle l+r+\ell_j+\iota-1/2\rangle_{r+\ell_j}a(m+l,-l-r-\ell_j)\\&\quad-(-1)^{\epsilon}\langle l+r-\iota-1/2\rangle_r\langle -m-l+\ell_i+\iota-1/2\rangle_{\ell_i}\tau(a)(-r-l,m+l-\ell_i)]\\&\quad=\sum_{l\in\mathcal{Z}}[\langle l+r+\ell_j+\iota-1/2\rangle_{r+\ell_j}a(m+l,-l-r-\ell_j)\\&\quad-(-1)^{\epsilon}\langle l+r+\iota-1/2\rangle_r\langle -m-l+\ell_i-\iota-1/2\rangle_{\ell_i}\tau(a)(-r-l-2\iota,m+l+2\iota-\ell_i)]\\&\quad=\sum_{l\in\mathcal{Z}}\langle l+r+\iota-1/2\rangle_r[\langle l+r+\ell_j+\iota-1/2\rangle_{\ell_j}a(m+l,-l-r-\ell_j)\\&\quad-(-1)^{\epsilon}\langle -m-l+\ell_i-\iota-1/2\rangle_{\ell_i}\tau(a)(-r-l-2\iota,m+l+2\iota-\ell_i)]\\&\quad=\sum_{l\in\mathcal{Z}}\langle l+r+\iota-1/2\rangle_r[(-1)^{\epsilon}\langle -l-r-\iota-1/2\rangle_{\ell_j}a(m+l,-l-r-\ell_j)\\&\quad-\langle m+l+\iota-1/2\rangle_{\ell_i}\tau(a)(-r-l-2\iota,m+l+2\iota-\ell_i)].\end{aligned}\tag{2.63}$$

Set

$$a_{\vec{\ell}}^{\tau,\iota}(l_1,l_2)=(-1)^{\epsilon}\langle l_2+\iota-1/2\rangle_{\ell_j}a(l_1,l_2+2\iota-\ell_j)-\langle l_1+\iota-1/2\rangle_{\ell_i}\tau(a)(l_2,l_1+2\iota-\ell_i)\tag{2.64}$$

for $a\in\mathcal{B}_{i,j}$ and $l_1,l_2\in\mathcal{Z}$. Then

$$a_{\vec{\ell}}^{\tau,\iota}(l_1,l_2)=-(-1)^{\epsilon}\tau(a)_{\vec{\ell}}^{\tau,\iota}(l_2,l_1).\tag{2.65}$$

Moreover, (2.61) becomes

$$\begin{aligned}\sigma_{\mathcal{M}}^{\iota}(a_{\vec{\ell}}^{\tau}(r,z))=&\sum_{l_1,l_2\in\mathcal{Z}}\langle -l_2-\iota-1/2\rangle_r a_{\vec{\ell}}^{\tau,\iota}(l_1,l_2)z^{-l_1-l_2-2\iota-r-1}+(r+\ell_j)!\Im_{0,r+\ell_j}(\mathrm{tr}\,a)\kappa_0\\&\times z^{-r-\ell_j-1}-(-1)^{\epsilon}r!\ell_i!\Im_{r,\ell_i}(\mathrm{tr}\,a)\kappa_0 z^{-r-\ell_i-1}.\end{aligned}\tag{2.66}$$

For $l,k\in\mathcal{Z}$ and $\ell\in\mathbb{N}$, we have

$$\langle l-1/2\rangle_{\ell}=(-1)^{\ell}\langle k-1/2\rangle_{\ell}\quad\text{if } l+k=\ell.\tag{2.67}$$

Given $a\in\mathcal{B}_{i_1,j_1}$, $b\in\mathcal{B}_{i_2,j_2}$ and $l_1,l_2,k_1,k_2\in\mathcal{Z}$, (2.23) and (2.64) imply

$$\begin{aligned}&[a_{\vec{\ell}}^{\tau,\iota}(l_1,l_2),b_{\vec{\ell}}^{\tau,\iota}(k_1,k_2)]\\&\quad=[(-1)^{\epsilon}\langle l_2+\iota-1/2\rangle_{\ell_{j_1}}a(l_1,l_2+2\iota-\ell_{j_1})-\langle l_1+\iota-1/2\rangle_{\ell_{i_1}}\tau(a)(l_2,l_1+2\iota-\ell_{i_1}),\\&\qquad(-1)^{\epsilon}\langle k_2+\iota-1/2\rangle_{\ell_{j_2}}b(k_1,k_2+2\iota-\ell_{j_2})-\langle k_1+\iota-1/2\rangle_{\ell_{i_2}}\tau(b)(k_2,k_1+2\iota-\ell_{i_2})]\\&\quad\equiv\langle k_1+\iota-1/2\rangle_{\ell_{i_2}}[\delta_{j_1,i_2}\delta_{l_2+k_1+2\iota,\ell_{j_1}}(ab)_{\vec{\ell}}^{\tau,\iota}(l_1,k_2)+\delta_{i_2,i_1}\delta_{l_1+k_1+2\iota,\ell_{j_1}}(\tau(b)a)_{\vec{\ell}}^{\tau,\iota}(k_2,l_2)]\\&\quad-\langle l_1+\iota-1/2\rangle_{\ell_{i_1}}[\delta_{j_2,i_1}\delta_{l_1+k_2+2\iota,\ell_{j_2}}(ba)_{\vec{\ell}}^{\tau,\iota}(k_1,l_2)+\delta_{i_1,i_2}\delta_{l_1+k_1+2\iota,\ell_{j_2}}(\tau(a)b)_{\vec{\ell}}^{\tau,\iota}(l_2,k_2)]\\&\qquad(\mathrm{mod}\ \mathbb{C}\kappa_0).\end{aligned}\tag{2.68}$$

We define

$$\check{\mathcal{A}}_{\vec{\ell}}^{\tau,\iota}=\mathrm{Span}\{a_{\vec{\ell}}^{\tau,\iota}(l_1,l_2)\mid i,j\in\overline{1,n};a\in\mathcal{B}_{i,j},l_1,l_2\in\mathcal{Z}\}+\mathbb{C}\kappa_0.\tag{2.69}$$

Then $\check{\mathcal{A}}_{\vec{\ell}}^{\tau,\iota}$ is a Lie subalgebra of $\check{\mathcal{A}}$ by (2.68). Set

$$\tilde{\ell}=|2\iota|+\max\{\ell_1,\ell_2,\ldots,\ell_n\}.\tag{2.70}$$

Define

$$\check{\mathcal{A}}_{\vec{\ell},m}^{\tau,\iota}=\mathrm{Span}\{a_{\vec{\ell}}^{\tau,\iota}(l_1,-l_2)\mid i,j\in\overline{1,n};a\in\mathcal{B}_{i,j},m<l_1,l_2\in\mathcal{Z}\}\quad\text{for }\tilde{\ell}<m\in\mathbb{N}.\tag{2.71}$$

By (2.23), (2.64), (2.67) and (2.68), $\check{\mathcal{A}}^{\tau,\iota}_{\vec{\ell},m}$ forms a Lie subalgebra of $\check{\mathcal{A}}^{\tau,\iota}_{\vec{\ell}}$.

Theorem 2.4. *Take $\iota \in \mathbb{Z}/2$. Let M be an $\check{\mathcal{A}}^{\tau,\iota}_{\vec{\ell}}$-module*

$$\text{generated by a subspace } M_0 \text{ such that } \check{\mathcal{A}}^{\tau,\iota}_{\vec{\ell},m}(M_0) = \{0\} \text{ for some } \tilde{\ell} < m \in \mathbb{N}. \tag{2.72}$$

Then we have the following representation σ_M of $\hat{\mathcal{A}}^{\tau}_{\vec{\ell}}$ on M: $\sigma_M(\kappa) = \kappa_0$ and

$$\sigma(a^{\tau}_{\vec{\ell}}(r,z)) = \sum_{l_1,l_2\in\mathcal{Z}} \langle -l_2 - \iota - 1/2\rangle_r a^{\tau,\iota}_{\vec{\ell}}(l_1,l_2) z^{-l_1-l_2-2\iota-r-1} + (r+\ell_j)!\Im_{0,r+\ell_j}(tr\,a)\kappa_0 \times z^{-r-\ell_j-1} - (-1)^{\epsilon} r!\ell_i!\Im_{r,\ell_i}(tr\,a)\kappa_0 z^{-r-\ell_i-1} \tag{2.73}$$

for $a \in \mathcal{B}_{i,j}$ and $r \in \mathbb{N}$.

Next we assume $\iota \in \mathbb{Z}$. When $\vec{\ell} \neq \vec{0}$, some of the coefficients $\langle -l_2 + \iota - 1/2\rangle_r$ in (2.73) would be zero and the related elements $a^{\tau,\iota}_{\vec{\ell}}(l_1,l_2)$ would yield obvious proper submodules. We want to exclude these elements $a^{\tau,\iota}_{\vec{\ell}}(l_1,l_2)$ and the obvious proper submodules caused by them in order to get irreducible representations later on. It can be verified that the subspace

$$\mathcal{I} = \mathrm{Span}\{a^{\tau,\iota}_{\vec{\ell}}(l_1-\iota, l_2-\iota) \mid i,j \in \overline{1,n};\, a \in \mathcal{B}_{i,j}, l_1,l_2 \in \mathcal{Z}, 0 < l_1 < \ell_i \text{ or } 0 < l_2 < \ell_j\} \tag{2.74}$$

forms an ideal of $\check{\mathcal{A}}^{\tau,\iota}_{\vec{\ell}}$.

Denote

$$\mathcal{Z}_i = \mathcal{Z} \setminus \{-\iota+1/2, -\iota+3/2\ldots, -\iota+\ell_i-1/2\} \quad \text{for } i \in \overline{1,n}. \tag{2.75}$$

Then the subspace

$$\check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell}} = \mathrm{Span}\{a^{\tau,\iota}_{\vec{\ell}}(l_1,l_2) \mid i,j \in \overline{1,n};\, a \in \mathcal{B}_{i,j}, l_1 \in \mathcal{Z}_i, l_2 \in \mathcal{Z}_j\} + \mathbb{C}\kappa_0 \tag{2.76}$$

forms a Lie subalgebra of $\check{\mathcal{A}}^{\tau,\iota}_{\vec{\ell}}$. Moreover,

$$\check{\mathcal{A}}^{\tau,\iota}_{\vec{\ell}} = \check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell}} \oplus \mathcal{I}. \tag{2.77}$$

Set

$$\check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell},m} = \mathrm{Span}\{a^{\tau,\iota}_{\vec{\ell}}(l_1,-l_2) \mid i,j \in \overline{1,n};\, a \in \mathcal{B}_{i,j}, m < l_1 \in \mathcal{Z}_i, m < l_2 \in \mathcal{Z}_j\} \tag{2.78}$$

for $\tilde{\ell} < m \in \mathbb{N}$. By (2.23), (2.64) and (2.68), $\check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell},m}$ forms a Lie subalgebra of $\check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell}}$. According to (2.73), we have:

Theorem 2.5. *Suppose $\iota \in \mathbb{Z}$. Let $\mathcal{N}$ be a $\check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell}}$-module*

$$\text{generated by a subspace } \mathcal{N}_0 \text{ such that } \check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell},m}(\mathcal{N}_0) = \{0\} \text{ for some } \tilde{\ell} < m \in \mathbb{N}. \tag{2.79}$$

Then we have the following representation $\sigma_{\mathcal{N}}$ of $\hat{\mathcal{A}}^{\tau}_{\vec{\ell}}$ on $\mathcal{N}$: $\sigma_{\mathcal{N}}(\kappa) = \kappa_0$ and

$$\sigma_{\mathcal{N}}(a^{\tau}_{\vec{\ell}}(r,z)) = \sum_{l_1\in\mathcal{Z}_i,l_2\in\mathcal{Z}_j} \langle -l_2 - \iota - 1/2\rangle_r a^{\tau,\iota}_{\vec{\ell}}(l_1,l_2) z^{-l_1-l_2-2\iota-r-1} + (r+\ell_j)!\Im_{0,r+\ell_j}(tr\,a)\kappa_0 \times z^{-r-\ell_j-1} - (-1)^{\epsilon} r!\ell_i!\Im_{r,\ell_i}(tr\,a)\kappa_0 z^{-r-\ell_i-1} \tag{2.80}$$

for $a \in \mathcal{B}_{i,j}$ and $r \in \mathbb{N}$.

Example 2.1. Let $k > 1$ be integer. The *Hecke algebra* $\mathcal{H}_k$ is an associative algebra generated by $\{T_1, \dots, T_{k-1}\}$ with the following defining relations

$$T_i T_j = T_j T_i \quad \text{whenever } |i - j| \geq 2, \tag{2.81}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i^2 = (q-1)T_i + q \tag{2.82}$$

for $i, j \in \overline{1, k-1}$, where $0 \neq q \in \mathbb{C}$. Let $\zeta \in \mathbb{C}$ be a fixed constant. According to Section 5 in [16], there exists a unique trace map "tr" of $\mathcal{H}_k$ such that

$$\mathrm{tr}\,(e) = \frac{1}{2}, \qquad \mathrm{tr}\,(aT_m b) = \zeta\,\mathrm{tr}\,(ab) \quad \text{for } a, b \in H_m \tag{2.83}$$

with $m \in \overline{1, k-2}$. This trace map is the key to define the well-known "Jones polynomials" of knots (e.g., cf. [16]).

We define a linear transformation τ on $\mathcal{H}_k$ by

$$\tau(T_{i_1} T_{i_2} \cdots T_{i_{r-1}} T_{i_r}) = T_{i_r} T_{i_{r-1}} \cdots T_{i_2} T_{i_1}. \tag{2.84}$$

Then τ satisfies (2.53) and (2.54) by (2.81)–(2.83) with a certain choice of $\{\mathcal{B}_{i,j}\}$, say $n = 1$ and $\mathcal{B}_{1,1} = \mathcal{H}_k$.

Example 2.2. Suppose that G is a group and 1 is its identity element. Let $\mathbb{C}[G]$ be the vector space with a basis $\{\varpi(g) \mid g \in \Gamma\}$, and multiplication:

$$\varpi(g_1) \cdot \varpi(g_2) = \varpi(g_1 g_2) \quad \text{for } g_1, g_2 \in \Gamma. \tag{2.85}$$

Then $\mathbb{C}[G]$ forms an associative algebra with the identity element $\varpi(1)$, which is called the *group algebra* of G. Define the "trace map" of $\mathbb{C}[G]$ by

$$\mathrm{tr}\,\varpi(g) = \delta_{1,g} \quad \text{for } g \in \Gamma \tag{2.86}$$

and a linear transformation τ on $\mathbb{C}[G]$ by

$$\tau(\varpi(g)) = \varpi(g^{-1}) \quad \text{for } g \in \Gamma. \tag{2.87}$$

It can be verified that τ satisfies (2.53) and (2.54) by (2.85)–(2.87) with a certain choice of $\{\mathcal{B}_{i,j}\}$, say $n = 1$ and $\mathcal{B}_{1,1} = \mathbb{C}[G]$.

The representations of $\hat{\mathcal{A}}_{\vec{\ell}}$ and $\hat{\mathcal{A}}_{\vec{\ell}}^{\tau}$ with $\mathcal{A} = \mathcal{H}_k, \mathbb{C}[G]$ and their related vertex algebras will be studied in our future works. In the rest of this paper, we will deal with the case when $\mathcal{A}$ is a matrix algebra.

3. Modules related to general linear algebras

In this section, we give detailed constructions of irreducible modules of the Lie algebras $\hat{\mathcal{A}}_{\vec{\ell}}$ in (2.48) and $\hat{\mathcal{A}}_{\vec{\ell}}^{\tau}$ in (2.59) when $\mathcal{A}$ is the $n \times n$ matrix algebra, from weighted irreducible modules of a central extension of the general linear Lie algebra of infinite matrices with finite number of nonzero entries.

Recall that $\overline{gl}(\infty)$ is a vector space with a basis $\{\mathcal{E}_{l,k} \mid l, k \in \mathcal{Z}\}$ and the multiplication:

$$\mathcal{E}_{l_1,l_2} \cdot \mathcal{E}_{k_1,k_2} = \delta_{l_2+k_1,0}\mathcal{E}_{l_1,k_2} \quad \text{for } l_1, l_2, k_1, k_2 \in \mathcal{Z}. \tag{3.1}$$

Moreover, $\overline{gl}(\infty)$ is isomorphic to the associative algebra of infinite matrices with finite number of nonzero entries. Let $M_{n\times n}(\mathbb{C})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{C}$. Then we have

$$M_{n\times n}(\mathbb{C}) \otimes_{\mathbb{C}} \overline{gl}(\infty) \cong \overline{gl}(\infty) \quad \text{as associative algebras.} \tag{3.2}$$

Take

$$\mathcal{A} = M_{n\times n}(\mathbb{C}) \quad \text{in the previous section} \tag{3.3}$$

with the sum of diagonal entries as the trace map "tr". Let $M_{n\times n}(\mathbb{A})$ be the algebra of $n \times n$ matrices with entries in $\mathbb{A}$ (cf. (2.1)–(2.3)). Again $E_{i,j}$ is the $n \times n$ matrix with 1 as its (i, j)-entry and 0 as the others. Recall the Lie algebra

$$\widehat{gl}(n, \mathbb{A}) = M_{n\times n}(\mathbb{A}) \oplus \mathbb{C}\kappa \tag{3.4}$$

with the Lie bracket:

$$[t^{m_1}\partial_t^{r_1}E_{i_1,j_1}+\mu_1\kappa, t^{m_2}\partial_t^{r_2}E_{i_2,j_2}+\mu_1\kappa] = \delta_{j_1,i_2}t^{m_1}\partial_t^{r_1}\cdot t^{m_2}\partial_t^{r_2}E_{i_1,j_2} - \delta_{i_1,j_2}t^{m_2}\partial_t^{r_2}\cdot t^{m_1}\partial_t^{r_1}E_{i_2,j_1} + (-1)^{r_1}\delta_{i_1,j_2}\delta_{j_1,i_2}\delta_{r_1+r_2,m_1+m_2}r_1!r_2!\binom{m_1}{r_1+r_2+1}\kappa \tag{3.5}$$

for $i,j\in\overline{1,n}$, $m_1,m_2\in\mathbb{Z}$, $r_1,r_2\in\mathbb{N}$ and $\mu_1,\mu_2\in\mathbb{C}$. The Lie algebras

$$\hat{\mathcal{A}}\cong\widehat{gl}(n,\mathbb{A}). \tag{3.6}$$

Fix an element $\vec{\ell}\in\mathbb{N}^n$. We have the Lie subalgebra

$$\widehat{gl}(\vec{\ell},\mathbb{A})=\sum_{i,j=1}^n\mathbb{A}\partial_t^{\ell_j}E_{i,j}+\mathbb{C}\kappa \tag{3.7}$$

of $\widehat{gl}(n,\mathbb{A})$ of type $\hat{\mathcal{A}}_{\vec{\ell}}$ (cf. (2.48)).

Let H be a toral Cartan subalgebra of a Lie algebra $\mathcal{G}$. We always denote

$$H^* = \text{the space of linear functions on } H. \tag{3.8}$$

A $\mathcal{G}$-module $\mathcal{M}$ is called *weighted* if

$$\mathcal{M}=\bigoplus_{\nu\in\mathcal{T}^*}\mathcal{M}_\nu,\qquad \mathcal{M}_\nu=\{u\in\mathcal{M}\mid h(u)=\nu(h)u, h\in H\}. \tag{3.9}$$

Recall the algebra

$$\tilde{gl}(\infty)=\overline{gl}(\infty)\oplus\mathbb{C}\kappa_0 \tag{3.10}$$

with the Lie bracket:

$$[\mathcal{E}_{l_1,l_2}+\mu_1\kappa_0,\mathcal{E}_{k_1,k_2}+\mu_2\kappa_0]=\mathcal{E}_{l_1,l_2}\mathcal{E}_{k_1,k_2}-\mathcal{E}_{k_1,k_2}\mathcal{E}_{l_1,l_2} + \delta_{l_1+k_2,0}\delta_{l_2+k_1,0}[H(l_1)H(l_2)-H(k_1)H(k_2)]\kappa_0 \tag{3.11}$$

for $l_1,l_2,k_1,k_2\in\mathcal{Z}$ and $\mu_1,\mu_2\in\mathbb{C}$ (cf. (2.23)), where κ_0 is a base element. By (2.21)–(2.23) and (3.2), the linear map

$$\kappa_0\leftrightarrow\kappa_0,\qquad \mathcal{E}_{i,j}(l+1/2,k-1/2)\leftrightarrow\mathcal{E}_{ln+i-1/2,kn-j+1/2}\quad\text{for } i,j\in\overline{1,n}, l,k\in\mathbb{Z}, \tag{3.12}$$

gives a Lie algebra isomorphism between the Lie algebra $\check{\mathcal{A}}$ and $\tilde{gl}(\infty)$ under our assumption (3.3). Moreover, the subspace

$$\mathcal{T}=\sum_{l\in\mathcal{Z}}\mathbb{C}\mathcal{E}_{l,-l}+\mathbb{C}\kappa_0 \tag{3.13}$$

forms a toral Cartan subalgebra of $\tilde{gl}(\infty)$. In fact, $\overline{gl}(\infty)$ and $\tilde{gl}(\infty)$ have a set of Chevalley generators satisfying the same Serre's defining relations. Thus they have exactly the same representation theory. We can take

$$\{\mathcal{E}_{l+1,-l}\mid l\in\mathcal{Z}\}\quad\text{as positive simple root vectors} \tag{3.14}$$

and

$$\{\mathcal{E}_{l,-l-1}\mid l\in\mathcal{Z}\}\quad\text{as negative simple root vectors.} \tag{3.15}$$

Furthermore,

$$[\mathcal{E}_{l+1,-l},\mathcal{E}_{l,-l-1}]=\mathcal{E}_{l+1,-l-1}-\mathcal{E}_{l,-l}\quad\text{for } -\frac{1}{2}\neq l\in\mathcal{Z} \tag{3.16}$$

and

$$[\mathcal{E}_{1/2,1/2}, \mathcal{E}_{-1/2,-1/2}] = \mathcal{E}_{1/2,-1/2} - \mathcal{E}_{-1/2,1/2} + \kappa_0. \tag{3.17}$$

Set

$$\tilde{gl}^m(\infty) = \mathrm{Span}\{\mathcal{E}_{l_1,-l_2}, \mathcal{E}_{-l_1,l_2} \mid m < l_1, l_2 \in \mathcal{Z}\} \quad \text{for } m \in \mathbb{N}. \tag{3.18}$$

Then (3.1) and (3.11) show that $\{\tilde{gl}^m(\infty) \mid m \in \mathbb{N}\}$ are Lie subalgebras of $\tilde{gl}(\infty)$. Let $\mathcal{M}$ be a weighted $\tilde{gl}(\infty)$-module

$$\text{generated by a subspace } \mathcal{M}_0 \quad \text{such that } \tilde{gl}^m(\infty)(\mathcal{M}_0) = \{0\} \quad \text{for some } m \in \mathbb{N}. \tag{3.19}$$

Fix a constant $\iota \in \mathbb{C}$. By Theorem 2.2 and (3.2), we have the following representation $\sigma^\iota_{\mathcal{M}}$ of $\widehat{gl}(\vec{\ell}, \mathbb{A})$: $\sigma^\iota_{\mathcal{M}}(\kappa) = \kappa_0$ and

$$\sigma^\iota_{\mathcal{M}}(t^m \partial_t^r E_{i,j}) = \sum_{l \in \mathbb{Z}} \langle \iota - l \rangle_r \mathcal{E}_{(m-r-l)n+i-1/2, ln-j+1/2} + r! \delta_{i,j} \delta_{r,m} \Im_{0,r} \kappa_0 \tag{3.20}$$

for $m \in \mathbb{Z}$, $r \in \mathbb{N}$ and $i, j \in \overline{1, n}$.

Theorem 3.1. *Suppose $\iota \notin \mathbb{Z}$. Then the representation $\sigma^\iota_{\mathcal{M}}$ of $\widehat{gl}(\vec{\ell}, \mathbb{A})$ is irreducible if and only if $\mathcal{M}$ is an irreducible $\tilde{gl}(\infty)$-module.*

Proof. Denote

$$h_{i,r} = \sigma^\iota_{\mathcal{M}}(t^r \partial_t^r E_{i,i}) = \sum_{l \in \mathbb{Z}} \langle l + \iota \rangle_r \mathcal{E}_{ln+i-1/2, -ln-i+1/2} + \Im_{0,r} \kappa_0 \tag{3.21}$$

for $i \in \overline{1, n}$ and $r \in \mathbb{N} + \ell_i$. Set

$$H = \sum_{i=1}^{n} \sum_{r=0}^{\infty} \mathbb{C} h_{i,r} \subset \mathrm{End}\, \mathcal{M}, \tag{3.22}$$

the space of linear transformations on $\mathcal{M}$. As operators on $\mathcal{M}$,

$$[h_{i,r}, \mathcal{E}_{ln+j_1-1/2, kn-j_2+1/2}] = [\delta_{i,j_1} \langle l + \iota \rangle_r - \delta_{i,j_2} \langle -k + \iota \rangle_r] \mathcal{E}_{ln+j_1-1/2, kn-j_2+1/2} \tag{3.23}$$

for $j_1, j_2 \in \overline{1, n}$ and $l, k \in \mathbb{Z}$. Using generating functions, we have:

$$\begin{aligned}
&\left[\sum_{r=0}^{\infty} \frac{x^r}{r!} h_{i,r+\ell_i}, \mathcal{E}_{ln+j_1-1/2, kn-j_2+1/2}\right] \\
&= \left[\delta_{i,j_1} \sum_{r=0}^{\infty} \frac{\langle l + \iota \rangle_{r+\ell_i} x^r}{r!} - \delta_{i,j_2} \sum_{r=0}^{\infty} \frac{\langle -k + \iota \rangle_{r+\ell_i} x^r}{r!}\right] \mathcal{E}_{ln+j_1-1/2, kn-j_2+1/2} \\
&= \left[\delta_{i,j_1} \langle l + \iota \rangle_{\ell_i} \sum_{r=0}^{\infty} \frac{\langle l + \iota - \ell_i \rangle_r x^r}{r!} - \delta_{i,j_2} \langle -k + \iota \rangle_{\ell_i} \sum_{r=0}^{\infty} \frac{\langle -k + \iota - \ell_i \rangle_r x^r}{r!}\right] \mathcal{E}_{ln+j_1-1/2, kn-j_2+1/2} \\
&= [\delta_{i,j_1} \langle l + \iota \rangle_{\ell_i} (x+1)^{l+\iota-\ell_i} - \delta_{i,j_2} \langle -k + \iota \rangle_{\ell_i} (x+1)^{-k+\iota-\ell_i}] \mathcal{E}_{ln+j_1-1/2, kn-j_2+1/2} \\
&= \frac{\mathrm{d}^{\ell_i}}{\mathrm{d}x^{\ell_i}} [\delta_{i,j_1} (x+1)^{l+\iota} - \delta_{i,j_2} (x+1)^{-k+\iota}] \mathcal{E}_{ln+j_1-1/2, kn-j_2+1/2}.
\end{aligned} \tag{3.24}$$

Note that when $\iota \notin \mathbb{Z}$,

$$\frac{\mathrm{d}^{\ell_i}}{\mathrm{d}x^{\ell_i}} (x+1)^{m_1+\iota} = \frac{\mathrm{d}^{\ell_i}}{\mathrm{d}x^{\ell_i}} (x+1)^{m_2+\iota} \quad \text{for } m_1, m_2 \in \mathbb{Z} \Longleftrightarrow m_1 = m_2 \tag{3.25}$$

and

$$\frac{\mathrm{d}^{\ell_i}}{\mathrm{d}x^{\ell_i}} [(x+1)^{l_1+\iota} - (x+1)^{-k_1+\iota}] = \frac{\mathrm{d}^{\ell_i}}{\mathrm{d}x^{\ell_i}} [(x+1)^{l_2+\iota} - (x+1)^{-k_2+\iota}] \tag{3.26}$$

$$\text{for } l_1, l_2, k_1, k_2 \in \mathbb{Z}, l_1 \neq -k_1 \iff l_1 = l_2, k_1 = k_2. \tag{3.27}$$

Denote by H^* the space of linear functions on H. Given $\rho \in H^*$, we set

$$\tilde{gl}(\infty)_{(\rho)} = \{\xi \in \tilde{gl}(\infty) \mid [h, \xi] = \rho(h)\xi \text{ for } h \in H\} \tag{3.28}$$

and

$$\mathcal{M}_{(\rho)} = \{w \in \mathcal{M} \mid h(w) = \rho(h)w \text{ for } h \in H\}. \tag{3.29}$$

Then

$$\tilde{gl}(\infty)_{(0)} = \mathcal{T}, \qquad \dim \tilde{gl}(\infty)_{(\rho)} = 1 \quad \text{for } 0 \neq \rho \in H^* \tag{3.30}$$

by (3.24)–(3.27). Moreover,

$$\tilde{gl}(\infty) = \bigoplus_{\rho \in H^*} \tilde{gl}(\infty)_{(\rho)}. \tag{3.31}$$

Since $\mathcal{M}$ is a weighted $\tilde{gl}(\infty)$-module, we have

$$\mathcal{M} = \bigoplus_{\rho \in H^*} \mathcal{M}_{(\rho)} \tag{3.32}$$

by (3.13), (3.21) and (3.22). If V is a $\tilde{gl}(\infty)$-submodule of $\mathcal{M}$, then V is a $\widehat{gl}(\vec{\ell}, \mathbb{A})$-submodule of $\mathcal{M}$ by (3.20). Suppose that U is a $\widehat{gl}(\vec{\ell}, \mathbb{A})$-submodule of $\mathcal{M}$. Then

$$U = \bigoplus_{\rho \in H^*} U_{(\rho)}, \qquad U_{(\rho)} = U \bigcap \mathcal{M}_{(\rho)}. \tag{3.33}$$

Since

$$\sigma^{\iota}_{\mathcal{M}}(t^m \partial_t^r E_{i,j})(U_{(\rho)}) = \sum_{l \in \mathbb{Z}} \langle \iota - l \rangle_r \mathcal{E}_{(m-r-l)n+i-1/2, ln-j+1/2}(U_{(\rho)}) + r! \delta_{i,j} \delta_{r,m} \Im_{0,r} \kappa_0(U_{(\rho)}) \subset U, \tag{3.34}$$

we have

$$\mathcal{E}_{(m-r-l)n+i-1/2, ln-j+1/2}(U_{(\rho)}) \subset U \tag{3.35}$$

for $m \in \mathbb{Z}, r \in \mathbb{N}, i, j \in \overline{1, n}$ such that $(m, i) \neq (0, j)$ by (3.30). Observe that $\{\mathcal{E}_{l_1, l_2} \mid l_1, l_2 \in \mathcal{Z}; l_1 \neq -l_2\}$ generates the Lie algebra $\tilde{gl}(\infty)$. Thus (3.34) implies

$$\tilde{gl}(\infty)(U_{(\rho)}) \subset U \quad \text{for } \rho \in H^*. \tag{3.36}$$

Therefore, U is a $\tilde{gl}(\infty)$-submodule. $\square$

The above theorem shows that we construct a family of irreducible representations $\{\sigma^{\iota}_{\mathcal{M}} \mid \iota \in \mathbb{C} \setminus \mathbb{Z}\}$ from any irreducible weighted $\tilde{gl}(\infty)$-module $\mathcal{M}$ satisfying (3.19). Set

$$E_{i,j}(r, z) = \sum_{m \in \mathbb{Z}} t^m \partial_t^r E_{i,j} z^{-m-1} \quad \text{for } i, j \in \overline{1, n}, \ r \in \mathbb{N}. \tag{3.37}$$

Recall that we obtained Theorem 2.3 from Theorem 2.2 by deleting those terms $\langle -l + \iota - 1/2 \rangle_r a(m - r - l, l)$ with $\langle -l + \iota - 1/2 \rangle_r = 0$ in (2.32) and partially shifting indices. The following theorem is obtained from Theorem 3.1 by deleting those terms $\langle \iota - l \rangle_r \mathcal{E}_{(m-r-l)n+i-1/2, ln-j+1/2}$ in (3.20) with $\langle \iota - l \rangle_r = 0$ and partially shifting indices. Ma [29] proved this when $\iota = 0$ in a more complicated form.

Theorem 3.2. *Suppose $\iota \in \mathbb{Z}$. Let $\mathcal{M}$ be a weighted $\tilde{gl}(\infty)$-module satisfying (3.19). We have the following representation $\sigma_{\vec{\ell}, \mathcal{M}}$ of $\widehat{gl}(\vec{\ell}, \mathbb{A})$: $\sigma_{\vec{\ell}, \mathcal{M}}(\kappa) = \kappa_0$ and*

$$\sigma_{\vec{\ell}, \mathcal{M}}(E_{i,j}(r, z)) = \sum_{l,k=0}^{n} [\langle -k - 1 \rangle_r \mathcal{E}_{(l-\iota)n+i-1/2, (k+\iota+1)n-j+1/2} z^{-l-k-\ell_i-r-2}$$

$$+ \langle k+\ell_j\rangle_r \mathcal{E}_{(l-\iota)n+i-1/2,(\iota-k)n-j+/2} z^{-l+k+\ell_j-\ell_i-r-1}$$
$$+ \langle k+\ell_j\rangle_r \mathcal{E}_{-(l+\iota+1)n+i-1/2,(\iota-k)n-j+1/2} z^{l+k+\ell_j-r}$$
$$+ \langle -k-1\rangle_r \mathcal{E}_{-(l+\iota+1)n+i-1/2,(k+\iota+1)n-j+1/2} z^{l-k-r-1}] + \delta_{i,j} r!\Im_{0,r}\kappa_0 z^{-r-1} \tag{3.38}$$

for $i, j \in \overline{1,n}$ and $r \in \mathbb{N}+\ell_j$. Moreover, $\sigma_{\vec{\ell},\mathcal{M}}$ is irreducible if and only if $\mathcal{M}$ is irreducible.

Denote

$$i^* = n+1-i \quad \text{for } i \in \overline{1,n}. \tag{3.39}$$

We fix $\epsilon \in \{0,1\}$ and take

$$\vec{\ell} = (\ell_1, \ell_2, \ldots, \ell_n) \in \mathbb{N}^n \quad \text{such that } \{\ell_1, \ell_2, \ldots, \ell_n\} \subset 2\mathbb{N}+\epsilon \tag{3.40}$$

and

$$\ell_i = \ell_{i^*} \quad \text{for } i \in \overline{1,n}. \tag{3.41}$$

For any

$$\sum_{i,j=1}^{n} \mu_{i,j} E_{i,j} \in \mathcal{A}, \tag{3.42}$$

we define

$$\left(\sum_{i,j=1}^{n} \mu_{i,j} E_{i,j}\right)^* = \sum_{i,j=1}^{n} \mu_{i,j} E_{j^*,i^*}. \tag{3.43}$$

Then $*$ is an involution of $\mathcal{A}$ (cf. (2.53) and (2.54)). Now the Lie algebra $\hat{\mathcal{A}}^*_{\vec{\ell}}$ (cf. (2.59)) becomes

$$\hat{o}(\vec{\ell}, \mathbb{A}) = \check{A}^*_{\vec{\ell}} = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m\in\mathbb{Z}} \mathbb{C}(t^m \partial_t^{r+\ell_j} E_{i,j} - (-1)^\epsilon (-\partial)^r t^m \partial_t^{\ell_i} E_{j^*,i^*}) + \mathbb{C}\kappa. \tag{3.44}$$

Again let $\mathcal{M}$ be a weighted $\tilde{gl}(\infty)$-module satisfying (3.19). Then we have the restricted representation of $\sigma^\iota_{\mathcal{M}}$ on $\hat{o}(\vec{\ell}, \mathbb{A})$ with $\sigma^\iota_{\mathcal{M}}(\kappa) = \kappa_0$ and

$$\sigma^\iota_{\mathcal{M}}(t^{m+r} \partial_t^{r+\ell_j} E_{i,j} - (-1)^\epsilon (-\partial)^r t^{m+r} \partial_t^{\ell_i} E_{j^*,i^*})$$
$$= \sum_{l\in\mathbb{Z}} [\langle l+\ell_j+\iota\rangle_{r+\ell_j} \mathcal{E}_{(m+l)n+i-1/2,(-l-\ell_j)n-j+1/2} - (-1)^\epsilon \langle l-\iota\rangle_r$$
$$\times \langle -m-l+\ell_i+\iota-1\rangle_{\ell_i} \mathcal{E}_{-ln-j+1/2,(m+l-\ell_i)n+i-1/2}]$$
$$+ [(r+\ell_i)!\Im_{0,r+\ell_i} - (-1)^\epsilon r!\ell_i!\Im_{r,\ell_i}]\delta_{m,\ell_i}\delta_{i,j}\kappa_0 \tag{3.45}$$

for $i, j \in \overline{1,n}$, $m \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Theorem 3.3. *Suppose $\iota \notin \mathbb{Z}/2$. Then the representation $\sigma^\iota_{\mathcal{M}}$ of $\hat{o}(\vec{\ell}, \mathbb{A})$ is irreducible if and only if $\mathcal{M}$ is an irreducible $\tilde{gl}(\infty)$-module.*

Proof. For $i \in \overline{1,n}$ and $r \in \mathbb{N}$, we define

$$\eta_{i,r} = \sigma^\iota_{\mathcal{M}}(t^{r+\ell_i} \partial_t^{r+\ell_i} E_{i,i} - (-1)^\epsilon (-\partial)^r t^{r+\ell_i} \partial_t^{\ell_i} E_{i^*,i^*})$$
$$= \sum_{l\in\mathbb{Z}} (\langle l+\iota\rangle_{r+\ell_i} \mathcal{E}_{ln+i-1/2,-ln-i+1/2} - \langle \ell_i-l-\iota\rangle_{r+\ell_i} \mathcal{E}_{ln-i+1/2,-ln+i-1/2})$$
$$+ [(r+\ell_i)!\Im_{0,r+\ell_i} - (-1)^\epsilon r!\ell_i!\Im_{r,\ell_i}]\kappa_0. \tag{3.46}$$

Set

$$H_o = \sum_{i=1}^{n}\sum_{r=0}^{\infty}\mathbb{C}\eta_{i,r} \subset \mathrm{End}\mathcal{M}, \tag{3.47}$$

the space of linear transformations on $\mathcal{M}$. As operators on $\mathcal{M}$,

$$\begin{aligned}[\eta_{i,r}, \mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2}] = [&\delta_{i,j_1}\langle l+\iota\rangle_{r+\ell_i} - \delta_{i^*,j_1}\langle \ell_i - l - 1 - \iota\rangle_{r+\ell_i} \\ &- \delta_{i,j_2}\langle \iota - k\rangle_{r+\ell_i} + \delta_{i^*,j_2}\langle k+\ell_i - 1 - \iota\rangle_{r+\ell_i}]\mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2}\end{aligned} \tag{3.48}$$

for $j_1, j_2 \in \overline{1,n}$ and $l, k \in \mathbb{Z}$. Using generating functions, we have:

$$\begin{aligned}&\left[\sum_{r=0}^{\infty}\frac{x^r}{r!}\eta_{i,r}, \mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2}\right] \\ &\quad= \frac{\mathrm{d}^{\ell_i}}{\mathrm{d}x^{\ell_i}}[\delta_{i,j_1}(x+1)^{l+\iota} - \delta_{i^*,j_1}(x+1)^{\ell_i-l-1-\iota} - \delta_{i,j_2}(x+1)^{\iota-k} \\ &\qquad+ \delta_{i^*,j_2}(x+1)^{k+\ell_i-1-\iota}]\mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2}.\end{aligned} \tag{3.49}$$

Since $\iota \notin \mathbb{Z}/2$, we have

$$l+\iota \neq k-\iota \quad \text{for } l,k\in\mathbb{Z}, i\in\overline{1,n}. \tag{3.50}$$

Thus

$$(x+1)^{l_1+\iota} - (x+1)^{k_1-\iota} = (x+1)^{l_2+\iota} - (x+1)^{k_2-\iota} \quad \text{for } l_1,l_2,k_1,k_2\in\mathbb{Z} \Longleftrightarrow l_1 = l_2, k_1 = k_2 \tag{3.51}$$

and

$$\begin{aligned}&(x+1)^{l_1+\iota} - (x+1)^{\ell_i-l_1-1-\iota} - (x+1)^{\iota-k_1} + (x+1)^{k_1+\ell_i-1-\iota} \\ &\quad= (x+1)^{l_2+\iota} - (x+1)^{\ell_i-l_2-1-\iota} - (x+1)^{\iota-k_2} + (x+1)^{k_2+\ell_i-1-\iota}\end{aligned} \tag{3.52}$$

for $l_1, k_1, l_2, k_2 \in \mathbb{Z}$ with $l_1 \neq -k_1, l_2 \neq -k_2$ if and only if $l_1 = l_2$ and $k_1 = k_2$. Denote by H_o^* the space of linear functions on H_o. Given $\rho \in H_o^*$, we set

$$\tilde{gl}(\infty)_{[\rho]} = \{\xi \in \tilde{gl}(\infty) \mid [h,\xi] = \rho(h)\xi \text{ for } h \in H_o\}. \tag{3.53}$$

Then

$$\tilde{gl}(\infty)_{[0]} = \mathcal{T}, \qquad \dim \tilde{gl}(\infty)_{[\rho]} = 1 \quad \text{for } 0 \neq \rho \in H_o^* \tag{3.54}$$

by the above arguments. Therefore, the conclusion follows the same arguments as those in (3.30)–(3.36). □

Next we suppose that

$$n = 2n_0 \text{ is an even positive integer.} \tag{3.55}$$

Moreover, we define the parity of indices:

$$p(i) = 0, \qquad p(n_0+i) = 1 \quad \text{for } i \in \overline{1,n_0}. \tag{3.56}$$

For any element in (3.42), we define

$$\left(\sum_{i,j=1}^{n}\mu_{i,j}E_{i,j}\right)^{\dagger} = \sum_{i,j=1}^{n}(-1)^{p(i)+p(j)}\mu_{i,j}E_{j^*,i^*}. \tag{3.57}$$

Then † is an involution of $\mathcal{A}$ (cf. (2.53) and (2.54)). Now the Lie algebra $\hat{\mathcal{A}}^{\dagger}_{\vec{\ell}}$ (cf. (2.59)) becomes

$$\widehat{sp}(\vec{\ell},\mathbb{A}) = \sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\sum_{m\in\mathbb{Z}}\mathbb{C}(t^m\partial_t^{r+\ell_j}E_{i,j} - (-1)^{p(i)+p(j)+\epsilon}(-\partial_t)^r t^m\partial_t^{\ell_i}E_{j^*,i^*}) + \mathbb{C}\kappa. \tag{3.58}$$

Let $\mathcal{M}$ be a weighted $\tilde{gl}(\infty)$-module satisfying (3.19). Then we have the restricted representation of $\sigma^{\iota}_{\mathcal{M}}$ on $\widehat{sp}(\vec{\ell}, \mathbb{A})$ with $\sigma^{\iota}_{\mathcal{M}}(\kappa) = \kappa_0$ and

$$\begin{aligned} &\sigma^{\iota}_{\mathcal{M}}(t^{m+r}\partial_t^{r+\ell_j} E_{i,j} - (-1)^{p(i)+p(j)+\epsilon}(-\partial)^r t^{m+r}\partial_t^{\ell_i} E_{j^*,i^*}) \\ &\quad = \sum_{l\in\mathbb{Z}} [\langle l+\ell_j+\iota\rangle_{r+\ell_j}\mathcal{E}_{(m+l)n+i-1/2,(-l-\ell_j)n-j+1/2} - (-1)^{p(i)+p(j)+\epsilon}\langle l-\iota\rangle_r \\ &\qquad \times \langle -m-l+\ell_i+\iota-1\rangle_{\ell_i}\mathcal{E}_{-ln-j+1/2,(m+l-\ell_i)n+i-1/2}] \\ &\qquad + [(r+\ell_i)!\Im_{0,r+\ell_i} - (-1)^{p(i)+p(j)+\epsilon} r!\ell_i!\Im_{r,\ell_i}]\delta_{i,j}\delta_{m,\ell_i}\kappa_0 \end{aligned} \tag{3.59}$$

for $i, j \in \overline{1,n}$, $m \in \mathbb{Z}$ and $r \in \mathbb{N}$.

Theorem 3.4. *Suppose $\iota \notin \mathbb{Z}/2$. Then the representation $\sigma^{\iota}_{\mathcal{M}}$ of $\widehat{sp}(\vec{\ell}, \mathbb{A})$ is irreducible if and only if $\mathcal{M}$ is an irreducible $\tilde{gl}(\infty)$-module.*

Example 3.1. Let $\lambda \in \mathcal{T}^*$ (cf. (3.13)) such that there exists a positive integer m_0 for which

$$\lambda(\mathcal{E}_{l,-l}) = \lambda(\mathcal{E}_{-l,l}) = 0 \quad \text{for } m_0 < l \in \mathcal{Z}. \tag{3.60}$$

Moreover, we set

$$\tilde{gl}(\infty)_+ = \mathrm{Span}\{\mathcal{E}_{l,k} \mid l,k \in \mathcal{Z}; l+k > 0\}, \qquad \tilde{gl}(\infty)_- = \mathrm{Span}\{\mathcal{E}_{l,k} \mid l,k \in \mathcal{Z}; l+k < 0\} \tag{3.61}$$

and

$$\tilde{gl}(\infty)_0 = \tilde{gl}(\infty)_+ + \mathcal{T} \tag{3.62}$$

(cf. (3.13)). Then $\tilde{gl}(\infty)_\pm$ and $\tilde{gl}(\infty)_0$ are Lie subalgebras of $\tilde{gl}(\infty)$. Define a one-dimensional $\tilde{gl}(\infty)_0$-module $\mathbb{C}v_\lambda$ by

$$\tilde{gl}(\infty)_+(v_\lambda) = \{0\}, \qquad h(v_\lambda) = \lambda(h)v_\lambda \quad \text{for } h \in \mathcal{T}. \tag{3.63}$$

Form an induced $\tilde{gl}(\infty)$-module

$$M_\lambda = U(\tilde{gl}(\infty)) \otimes_{U(\tilde{gl}(\infty)_0)} \mathbb{C}v_\lambda \cong U(\tilde{gl}(\infty)_-) \otimes_{\mathbb{C}} \mathbb{C}v_\lambda, \tag{3.64}$$

that is, a Verma module. It is known that M_λ has a unique maximal proper submodule N_λ. Thus

$$\mathcal{M} = M_\lambda / N_\lambda \tag{3.65}$$

is a weighted irreducible $\tilde{gl}(\infty)$-module satisfying (3.19). Suppose

$$\lambda_l = \lambda(\mathcal{E}_{l+1,-l-1} - \mathcal{E}_{l,-l}), \qquad \lambda_{-1/2} = \lambda(\mathcal{E}_{1/2,-1/2} - \mathcal{E}_{-1/2,1/2} + \kappa_0) \in \mathbb{N} \tag{3.66}$$

for $-1/2 \neq l \in \mathcal{Z}$. Then

$$N_\lambda = \sum_{r\in\mathcal{Z}} U(\tilde{gl}(\infty)_-)\mathcal{E}_{r,-r-1}^{\lambda_r+1} \otimes v_\lambda \tag{3.67}$$

by the structure of the maximal proper submodule of a Verma module with dominant integral highest weight (e.g. cf. [21]).

Example 3.2. Let m be a fixed positive integer. Set

$$I = \{i-1/2, -i+1/2 \mid i \in \overline{1,m}\}, \qquad J = \mathcal{Z} \setminus I. \tag{3.68}$$

The subspaces

$$\mathcal{G} = \mathrm{Span}\{\mathcal{E}_{l,k}, \kappa_0 \mid l,k \in I\} \quad \text{and} \quad \overline{\mathcal{G}} = \mathrm{Span}\{\mathcal{E}_{l,k}, \kappa_0 \mid l,k \in J\} \tag{3.69}$$

form Lie subalgebras of $\tilde{gl}(\infty)$. The algebra $\mathcal{G}$ is isomorphic to the one-dimensional central extension of $gl(2m,\mathbb{C})$ with

$$H=\sum_{l\in I}\mathbb{C}\mathcal{E}_{l,-l}+\mathbb{C}\kappa_0 \tag{3.70}$$

as a Cartan subalgebra. Moreover, the algebra $\overline{\mathcal{G}}$ is isomorphic to $\tilde{gl}(\infty)$ with

$$\overline{H}=\sum_{l\in J}\mathbb{C}\mathcal{E}_{l,-l}+\mathbb{C}\kappa_0 \tag{3.71}$$

as a Cartan subalgebra.

Denote

$$\mathcal{L}_0=\mathcal{G}+\overline{\mathcal{G}},\ \ \mathcal{L}_-=\operatorname{Span}\{\mathcal{E}_{l,r}\mid l\in J,r\in I\},\qquad \mathcal{L}_+=\operatorname{Span}\{\mathcal{E}_{r,l}\mid r\in I,l\in J\}. \tag{3.72}$$

Then $\mathcal{L}_0$ and $\mathcal{L}_\pm$ are Lie subalgebras of $\tilde{gl}(\infty)$. In fact,

$$\tilde{gl}(\infty)=\mathcal{L}_-\oplus\mathcal{L}_0\oplus\mathcal{L}_+. \tag{3.73}$$

Note that

$$\overline{\mathcal{G}}_\pm=\overline{\mathcal{G}}\bigcap\tilde{gl}(\infty)_\pm \tag{3.74}$$

(cf. (3.61)) are Lie subalgebras of $\overline{\mathcal{G}}$ and

$$\overline{\mathcal{G}}=\overline{\mathcal{G}}_-\oplus\overline{H}\oplus\overline{\mathcal{G}}_-. \tag{3.75}$$

In particular, we have a Borel subalgebra

$$\overline{\mathcal{B}}=\overline{H}+\overline{\mathcal{G}}_+. \tag{3.76}$$

Take any weighted irreducible $\mathcal{G}$-module M_0, which may not necessarily be of the highest-weight type. We extend the action of κ_0 to that of $\overline{\mathcal{B}}$ by

$$\overline{\mathcal{G}}_+(M_0)=\mathcal{E}_{k,-k}(M_0)=\{0\}\quad\text{for } k\in J. \tag{3.77}$$

Form an induced $\overline{\mathcal{G}}$-module

$$M_1=U(\overline{\mathcal{G}})\otimes_{U(\overline{\mathcal{B}})}M_0\cong U(\overline{\mathcal{G}}_-)\otimes_{\mathbb{C}}M_0. \tag{3.78}$$

Since

$$[\mathcal{G},\overline{\mathcal{G}}]=\{0\}, \tag{3.79}$$

M_1 becomes an $\mathcal{L}_0$-module by letting $\mathcal{G}$ act on the second factor. Moreover, M_1 has a unique maximal proper $\mathcal{L}_0$-submodule M_2. Form a quotient $\mathcal{L}_0$-module

$$\mathcal{M}_0=M_1/M_2. \tag{3.80}$$

In fact, (3.77) yields

$$\mathcal{E}_{l,-k}(M_0+M_2)=\mathcal{E}_{-l,k}(M_0+M_2)\subset M_2\quad\text{for } m<l,k\in\mathcal{Z}. \tag{3.81}$$

Note that

$$[\mathcal{L}_0,\mathcal{L}_\pm]\subset\mathcal{L}_\pm. \tag{3.82}$$

So

$$\mathcal{L}'=\mathcal{L}_0+\mathcal{L}_+ \tag{3.83}$$

forms a Lie subalgebra of $\tilde{gl}(\infty)$. We extend an action of $\mathcal{L}'$ on $\mathcal{M}_0$ from that of $\mathcal{L}_0$ by

$$\mathcal{L}_+(\mathcal{M}_0) = \{0\}. \tag{3.84}$$

The expression (3.82) implies that $\mathcal{M}_0$ becomes an $\mathcal{L}'$-module. Form an induced $\tilde{gl}(\infty)$-module:

$$\mathcal{M}_1 = U(\tilde{gl}(\infty)) \otimes_{U(\mathcal{L}')} \mathcal{M}_0 \cong U(\mathcal{L}_-) \otimes_{\mathbb{C}} \mathcal{M}_0. \tag{3.85}$$

It can be verified that $\mathcal{M}_1$ has a unique maximal proper $\tilde{gl}(\infty)$-submodule $\mathcal{M}_2$. The quotient

$$\mathcal{M} = \mathcal{M}_1/\mathcal{M}_2 \tag{3.86}$$

is a weighted irreducible $\tilde{gl}(\infty)$-module satisfying (3.19) by (3.81).

4. Modules with $\iota \in \mathbb{Z} + 1/2$ related to skew elements

In this section, we give detailed constructions of irreducible modules of the Lie algebras $\hat{\mathcal{A}}_{\vec{\ell}}^{\tau}$ in (2.59) with $\iota \in \mathbb{Z} + 1/2$ when $\mathcal{A}$ is the $n \times n$ matrix algebra, from weighted irreducible modules of central extensions of the Lie algebras of infinite skew matrices with finite number of nonzero entries.

Recall the Lie algebra $\overline{gl}(\infty)$ defined in (3.1). The subspaces

$$\bar{o}_d(\infty) = \sum_{k,l\in\mathcal{Z}} \mathbb{C}(\mathcal{E}_{l,k} - \mathcal{E}_{k,l}), \tag{4.1}$$

$$\bar{o}_b(\infty) = \sum_{k,l\in\mathcal{Z}} \mathbb{C}(\mathcal{E}_{l,k} - \mathcal{E}_{k-1,l+1}), \tag{4.2}$$

and

$$\overline{sp}(\infty) = \sum_{k,l\in\mathcal{Z};kl<0} \mathbb{C}(\mathcal{E}_{k,l} - \mathcal{E}_{l,k}) + \sum_{k,l\in\mathcal{Z};kl>0} \mathbb{C}(\mathcal{E}_{k,l} + \mathcal{E}_{l,k}) \tag{4.3}$$

form Lie subalgebra skew elements in $\overline{gl}(\infty)$. Take

$$\{\mathcal{E}_{l+1,-l} - \mathcal{E}_{-l,l+1}, \mathcal{E}_{3/2,1/2} - \mathcal{E}_{1/2,3/2} \mid 0 < l \in \mathcal{Z}\} \tag{4.4}$$

as positive simple root vectors of $\bar{o}_d(\infty)$ and

$$\{\mathcal{E}_{l,-l-1} - \mathcal{E}_{-l-1,l}, \mathcal{E}_{-1/2,-3/2} - \mathcal{E}_{-3/2,-1/2} \mid 0 < l \in \mathcal{Z}\} \tag{4.5}$$

as negative simple root vectors of $\bar{o}_d(\infty)$. Choose

$$\{\mathcal{E}_{l,1-l} - \mathcal{E}_{-l,l+1} \mid 0 < l \in \mathcal{Z}\} \tag{4.6}$$

as positive simple root vectors of $\bar{o}_b(\infty)$ and

$$\{\mathcal{E}_{l-1,-l} - \mathcal{E}_{-l-1,l} \mid 0 < l \in \mathcal{Z}\} \tag{4.7}$$

as negative simple root vectors of $\bar{o}_b(\infty)$. Pick

$$\{\mathcal{E}_{l+1,-l} - \mathcal{E}_{-l,l+1}, \mathcal{E}_{1/2,1/2} \mid 0 < l \in \mathcal{Z}\} \tag{4.8}$$

as positive simple root vectors of $\overline{sp}(\infty)$ and

$$\{\mathcal{E}_{l,-l-1} - \mathcal{E}_{-l-1,l}, \mathcal{E}_{-1/2,-1/2} \mid 0 < l \in \mathcal{Z}\} \tag{4.9}$$

as negative simple root vectors of $\overline{sp}(\infty)$.

Again we assume $\mathcal{A} = M_{n\times n}(\mathbb{C})$ in the settings of Section 2. Recall the Lie algebra $\check{\mathcal{A}}_{\vec{\ell}}^{\tau,\iota}$ defined in (2.69), the assumption (3.40), the involution $*$ defined in (3.43) and the involution $\dagger$ defined in (3.57).

Theorem 4.1. *We have the following Lie algebra isomorphisms:*

$$\check{\mathcal{A}}_{\vec{\ell}}^{*,\iota}/\mathbb{C}\kappa_0 \cong \begin{cases} \bar{o}_b(\infty) & \text{if } \epsilon = 0, n \in 2\mathbb{N}+1, \\ \bar{o}_d(\infty) & \text{if } \epsilon = 0, n \in 2\mathbb{N}, \\ \overline{sp}(\infty) & \text{if } \epsilon = 1 \end{cases} \tag{4.10}$$

and

$$\check{\mathcal{A}}_{\vec{\ell}}^{\dagger,\iota}/\mathbb{C}\kappa_0 \cong \begin{cases} \overline{sp}(\infty) & \text{if } \epsilon = 0, \\ \bar{o}_d(\infty) & \text{if } \epsilon = 1 \end{cases} \tag{4.11}$$

if n is even.

Proof. We write

$$\iota = \iota_0 + 1/2, \qquad \iota_0 \in \mathbb{Z}. \tag{4.12}$$

By assumptions (3.40) and (3.41),

$$\ell_i = \ell_{i^*} = 2m_i + \epsilon \quad \text{with } m_i \in \mathbb{N} \quad \text{for } i \in \overline{1,n}. \tag{4.13}$$

Thus (2.64) and (2.69) give

$$\check{\mathcal{A}}_{\vec{\ell}}^{*,\iota} = \sum_{i,j=1}^{n} \sum_{l,k\in\mathcal{Z}} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-1\rangle_{\ell_j} E_{i,j}(l+m_i-\iota_0, k+\iota_0-m_j) - \langle l+m_i\rangle_{\ell_i} E_{j^*,i^*}(k+m_j-\iota_0+\epsilon-1, l-m_i+\iota_0+1-\epsilon)) + \mathbb{C}\kappa_0, \tag{4.14}$$

and if $n = 2n_0$ is even,

$$\check{\mathcal{A}}_{\vec{\ell}}^{\dagger,\iota} = \sum_{i,j=1}^{n} \sum_{l,k\in\mathcal{Z}} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-1\rangle_{\ell_j} E_{i,j}(l+m_i-\iota_0, k+\iota_0-m_j) - (-1)^{p(i)+p(j)} \times \langle l+m_i\rangle_{\ell_i} E_{j^*,i^*}(k+m_j-\iota_0+\epsilon-1, l+\iota_0-m_i+1-\epsilon)) + \mathbb{C}\kappa_0 \tag{4.15}$$

(cf. (3.56)). According to (3.12), we set

$$\mathcal{G}_{\epsilon}^{*,\vec{m}} = \sum_{i,j=1}^{n} \sum_{l,k\in\mathbb{Z}} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j} \mathcal{E}_{(l+m_i-\iota_0)n+i-1/2,(k-m_j+\iota_0)n-j+1/2} - \langle l+m_i+1/2\rangle_{\ell_i} \mathcal{E}_{(k+m_j-\iota_0+\epsilon-1)n-j+1/2,(l-m_i+\iota_0+1-\epsilon)n+i-1/2}) + \mathbb{C}\kappa_0, \tag{4.16}$$

and

$$\mathcal{G}_{\epsilon}^{\dagger,\vec{m}} = \sum_{i,j=1}^{n} \sum_{l,k\in\mathbb{Z}} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j} \mathcal{E}_{(l+m_i-\iota_0)n+i-1/2,(k-m_j+\iota_0)n-j+1/2} - (-1)^{p(i)+p(j)}\langle l+m_i+1/2\rangle_{\ell_i} \mathcal{E}_{(k+m_j-\iota_0+\epsilon-1)n-j+1/2,(l-m_i+\iota_0+1-\epsilon)n+i-1/2}) + \mathbb{C}\kappa_0 \tag{4.17}$$

if n is even. Then $\mathcal{G}_{\epsilon}^{*,\vec{m}}$ and $\mathcal{G}_{\epsilon}^{\dagger,\vec{m}}$ are Lie subalgebras of $\tilde{gl}(\infty)$. Moreover,

$$\check{\mathcal{A}}_{\vec{\ell}}^{*,\iota} \cong \mathcal{G}_{\epsilon}^{*,\vec{m}}, \qquad \check{\mathcal{A}}_{\vec{\ell}}^{\dagger,\iota} \cong \mathcal{G}_{\epsilon}^{\dagger,\vec{m}} \tag{4.18}$$

by (3.12).

Define

$$\bar{\mathcal{G}}_{\epsilon}^{*,\vec{m}} = \sum_{i,j=1}^{n} \sum_{l,k\in\mathbb{Z}} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,kn-j+1/2} - \langle l+m_i+1/2\rangle_{\ell_i} \mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) \tag{4.19}$$

and

$$\begin{aligned}\bar{\mathcal{G}}_{\epsilon}^{\dagger,\vec{m}} = \sum_{i,j=1}^{n}\sum_{l,k\in\mathbb{Z}} & \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} \\ & -(-1)^{p(i)+p(j)}\langle l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2})\end{aligned} \tag{4.20}$$

if n is even. Then $\bar{\mathcal{G}}_{\epsilon}^{*,\vec{m}}$ and $\bar{\mathcal{G}}_{\epsilon}^{\dagger,\vec{m}}$ are Lie subalgebras of $\overline{gl}(\infty)$. The conclusion follows from (4.18) and the following facts:

$$\mathcal{G}_0^{*,\vec{m}}/\mathbb{C}\kappa_0 \cong \bar{\mathcal{G}}_0^{*,\vec{m}} \cong \bar{o}_b(\infty) \quad \text{if } n \text{ is odd,} \tag{4.21}$$

$$\mathcal{G}_0^{*,\vec{m}}/\mathbb{C}\kappa_0 \cong \bar{\mathcal{G}}_0^{*,\vec{m}} \cong \bar{o}_d(\infty) \quad \text{if } n \text{ is even,} \tag{4.22}$$

$$\mathcal{G}_1^{*,\vec{m}}/\mathbb{C}\kappa_0 \cong \bar{\mathcal{G}}_1^{*,\vec{m}} \cong \overline{sp}(\infty), \qquad \mathcal{G}_0^{\dagger,\vec{m}}/\mathbb{C}\kappa_0 \cong \bar{\mathcal{G}}_0^{\dagger,\vec{m}} \cong \overline{sp}(\infty), \tag{4.23}$$

$$\mathcal{G}_1^{\dagger,\vec{m}}/\mathbb{C}\kappa_0 \cong \bar{\mathcal{G}}_1^{\dagger,\vec{m}} \cong \bar{o}_d(\infty). \quad \square \tag{4.24}$$

Next we want to study highest-weight irreducible modules. Isomorphisms in (4.21)–(4.24) motivate us to adjust the definition of the Lie algebra $\tilde{gl}(\infty)$ (cf. (3.10) and (3.11)) by modifying the coefficient of κ_0 in (3.11). Moreover, for any $l \in \mathbb{Z}$, we write

$$l = l_Q n + l_R, \quad l_Q \in \mathbb{Z}, l_R \in \overline{1,n}. \tag{4.25}$$

Let $\vec{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$ and define a map $\alpha_{\vec{m}}^{\iota_0} : \mathcal{Z}^4 \to \mathbb{C}$ by:

$$\begin{aligned}\alpha_{\vec{m}}^{\iota_0}(l_1, l_2; k_1, k_2) = {} & H(l_1 + (m_{(l_1+1/2)_R} - \iota_0)n)H(l_2 + (\iota_0 - m_{(-l_2+1/2)_R})n) \\ & - H(k_1 + (m_{(k_1+1/2)_R} - \iota_0)n)H(k_2 + (\iota_0 - m_{(-k_2+1/2)_R})n)\delta_{l_1+k_2,0}\delta_{l_2+k_1,0}\end{aligned} \tag{4.26}$$

for $l_1, l_2, k_1, k_2 \in \mathcal{Z}$ (cf. (1.15)). Set

$$\tilde{gl}_{\vec{m}}^{\iota_0}(\infty) = \overline{gl}(\infty) \oplus \mathbb{C}\kappa_0 \tag{4.27}$$

(cf. (3.1)), where κ_0 is a base element. We have the following Lie bracket on $\tilde{gl}_{\vec{m}}^{\iota_0}(\infty)$:

$$[\mathcal{E}_{l_1,l_2} + \mu_1\kappa_0, \mathcal{E}_{k_1,k_2} + \mu_2\kappa_0] = \mathcal{E}_{l_1,l_2}\mathcal{E}_{k_1,k_2} - \mathcal{E}_{k_1,k_2}\mathcal{E}_{l_1,l_2} + \alpha_{\vec{m}}^{\iota_0}(l_1, l_2; k_1, k_2)\kappa_0 \tag{4.28}$$

for $l_1, l_2, k_1, k_2 \in \mathcal{Z}$. In particular, the Lie algebras

$$\tilde{gl}_{\vec{0}}^{0}(\infty) \cong \tilde{gl}(\infty). \tag{4.29}$$

Assume that (4.13) holds. Now we define

$$\begin{aligned}\mathcal{L}_{\iota_0,\epsilon}^{*,\vec{m}} = \sum_{i,j=1}^{n}\sum_{l,k\in\mathbb{Z}} & \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} \\ & -\langle l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) + \mathbb{C}\kappa_0,\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}\mathcal{L}_{\iota_0,\epsilon}^{\dagger,\vec{m}} = \sum_{i,j=1}^{n}\sum_{l,k\in\mathbb{Z}} & \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} - (-1)^{p(i)+p(j)} \\ & \times\langle l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) + \mathbb{C}\kappa_0\end{aligned} \tag{4.31}$$

if n is even. Then $\mathcal{L}_{\iota_0,\epsilon}^{*,\vec{m}}$ and $\mathcal{L}_{\iota_0,\epsilon}^{\dagger,\vec{m}}$ are Lie subalgebras of $\tilde{gl}_{\vec{m}}^{\iota_0}(\infty)$. Moreover,

$$\check{\mathcal{A}}_{\vec{\ell}}^{*,\iota} \cong \mathcal{G}_{\epsilon}^{*,\vec{m}} \cong \mathcal{L}_{\iota_0,\epsilon}^{*,\vec{m}}, \qquad \check{\mathcal{A}}_{\vec{\ell}}^{\dagger,\iota} \cong \mathcal{G}_{\epsilon}^{\dagger,\vec{m}} \cong \mathcal{L}_{\iota_0,\epsilon}^{\dagger,\vec{m}}. \tag{4.32}$$

Write

$$n = 2n_0 + \varepsilon, \quad n_0 \in \mathbb{N}, \varepsilon \in \{0, 1\}. \tag{4.33}$$

We denote

$$\begin{aligned} e^*_{\epsilon,l} = & \frac{1}{\langle m_{l_R} - l_Q + \epsilon - 3/2\rangle_{\ell_{l_R}} \langle m_{(l+1)_R} + (l+1)_Q + 1/2\rangle_{\ell_{(l+1)_R}}} \\ & \times [(-1)^\epsilon \langle m_{l_R} - l_Q + \epsilon - 3/2\rangle_{\ell_{l_R}} \mathcal{E}_{l+1/2,-l+1/2} \\ & - \langle m_{(l+1)_R} + (l+1)_Q + 1/2\rangle_{\ell_{(l+1)_R}} \mathcal{E}_{-l+(\epsilon-1)n+1/2, l+(1-\epsilon)n+1/2}] \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} f^*_{\epsilon,l} = & (-1)^\epsilon \langle m_{(l+1)_R} - (l+1)_Q + \epsilon - 3/2\rangle_{\ell_{(l+1)_R}} \mathcal{E}_{l-1/2,-l-1/2} \\ & - \langle m_{l_R} + l_Q + 1/2\rangle_{\ell_{l_R}} \mathcal{E}_{-l+(\epsilon-1)n-1/2, l+(1-\epsilon)n-1/2} \end{aligned} \tag{4.35}$$

for $l \in \mathbb{N} + \delta_{\epsilon,0}(\delta_{\varepsilon,0} - n_0) + \delta_{\epsilon,1}$. When $\epsilon = \varepsilon = 0$, we define

$$\begin{aligned} e^*_{0,-n_0} = & \frac{1}{\langle m_{n_0} - 1/2\rangle_{\ell_{n_0}} \langle m_{(n_0+2)_R} + (n_0+2)_Q - 1/2\rangle_{\ell_{(n_0+2)_R}}} \\ & \times [\langle m_{n_0} - 1/2\rangle_{\ell_{n_0}} \mathcal{E}_{-n_0+3/2, n_0+1/2} - \langle m_{(n_0+2)_R} + (n_0+2)_Q - 1/2\rangle_{\ell_{(n_0+2)_R}} \mathcal{E}_{-n_0+1/2, n_0+3/2}] \end{aligned} \tag{4.36}$$

and

$$f^*_{0,-n_0} = \langle m_{(n_0+2)_R} - (n_0+2)_Q - 1/2\rangle_{\ell_{(n_0+2)_R}} \mathcal{E}_{-n_0-1/2, n_0-3/2} - \langle m_{n_0} - 1/2\rangle_{\ell_{n_0}} \mathcal{E}_{-n_0-3/2, n_0-1/2}. \tag{4.37}$$

Furthermore, we let

$$e^*_{1,0} = \mathcal{E}_{1/2,1/2}, \qquad f^*_{1,0} = \mathcal{E}_{-1/2,-1/2}. \tag{4.38}$$

Suppose $\varepsilon = 0$. We define

$$\begin{aligned} e^\dagger_{\epsilon,l} = & \frac{1}{\langle m_{l_R} - l_Q + \epsilon - 3/2\rangle_{\ell_{l_R}} \langle m_{(l+1)_R} + (l+1)_Q + 1/2\rangle_{\ell_{(l+1)_R}}} \\ & \times [(-1)^\epsilon \langle m_{l_R} - l_Q + \epsilon - 3/2\rangle_{\ell_{l_R}} \mathcal{E}_{l+1/2,-l+1/2} - (-1)^{p((l+1)_R)+p(l_R)} \\ & \times \langle m_{(l+1)_R} + (l+1)_Q + 1/2\rangle_{\ell_{(l+1)_R}} \mathcal{E}_{-l+(\epsilon-1)n+1/2, l+(1-\epsilon)n+1/2}], \end{aligned} \tag{4.39}$$

and

$$\begin{aligned} f^\dagger_{\epsilon,l} = & (-1)^\epsilon \langle m_{(l+1)_R} - (l+1)_Q + \epsilon - 3/2\rangle_{\ell_{(l+1)_R}} \mathcal{E}_{l-1/2,-l-1/2} \\ & - (-1)^{p((l+1)_R)+p(l_R)} \langle m_{l_R} + l_Q + 1/2\rangle_{\ell_{l_R}} \mathcal{E}_{-l+(\epsilon-1)n-1/2, l+(1-\epsilon)n-1/2} \end{aligned} \tag{4.40}$$

for $l \in \mathbb{N} + 1 - n_0\delta_{\epsilon,0}$. Moreover, we set

$$e^\dagger_{0,-n_0} = \mathcal{E}_{-n_0+1/2, n_0+1/2}, \qquad f^\dagger_{0,0} = \mathcal{E}_{-n_0-1/2, n_0-1/2}. \tag{4.41}$$

Furthermore, we let

$$\begin{aligned} e^\dagger_{1,,0} = & \frac{1}{\langle m_1 + 1/2\rangle_{\ell_1} \langle m_{2_R} + 2_Q + 1/2\rangle_{\ell_{2_R}}} \\ & \times [\langle m_1 + 1/2\rangle_{\ell_1} \mathcal{E}_{3/2,1/2} + (-1)^{p(n)+p(2)} \langle m_{2_R} + 2_Q + 1/2\rangle_{\ell_{2_R}} \mathcal{E}_{1/2,3/2}] \end{aligned} \tag{4.42}$$

and

$$f^\dagger_{1,,0} = \langle m_{2_R} - 2_Q - 1/2\rangle_{\ell_{2_R}} \mathcal{E}_{-1/2,-3/2} + (-1)^{p(n)+p(2)} \langle m_1 + 3/2\rangle_{\ell_1} \mathcal{E}_{-3/2,-1/2}. \tag{4.43}$$

For convenience, we always assume

$$\tau \in \{*, \dagger\}. \tag{4.44}$$

Under the above settings,

$$\{e^{\tau}_{\epsilon,l} \mid l \in \mathbb{N} - \delta_{0,\epsilon} n_0\} \quad \text{is the set of positive simple root vectors of } \mathcal{L}^{\tau,\vec{\ell}}_{\iota_0,\epsilon} \tag{4.45}$$

and

$$\{f^{\tau}_{\epsilon,l} \mid l \in \mathbb{N} - \delta_{0,\epsilon} n_0\} \text{ is the set of negative simple root vectors of } \mathcal{L}^{\tau,\vec{m}}_{\iota,\epsilon}. \tag{4.46}$$

Set

$$\vartheta^{\epsilon}_l = \mathcal{E}_{l-1/2,-l+1/2} - \mathcal{E}_{-l+(\epsilon-1)n+1/2,l+(1-\epsilon)n-1/2} \tag{4.47}$$

for $l \in \mathbb{N} - n_0\delta_{\epsilon,0}$. Define

$$\begin{aligned}\omega^{\tau}_{\epsilon,l} = {} & \alpha^{\iota_0}_{\vec{m}}(l+1/2, -l+1/2; l-1/2, -l-1/2) + \alpha^{\iota_0}_{\vec{m}}(-l+(\epsilon-1)n+1/2, \\ & l+(1-\epsilon)n+1/2; -l+(\epsilon-1)n-1/2, l+(1-\epsilon)n-1/2)\end{aligned} \tag{4.48}$$

for $l \in \mathbb{N} + \delta_{\epsilon,0}(\delta_{\varepsilon,0} - n_0) + \delta_{\epsilon,1}$ if $\tau = *$, and $l \in \mathbb{N} + 1 - n_0\delta_{\epsilon,0}$ when $\tau = \dagger$. Moreover, we let

$$\begin{aligned}\omega^{*}_{0,-n_0} = {} & \alpha^{\iota_0}_{\vec{m}}(-n_0+3/2, n_0+1/2; -n_0-1/2, n_0-3/2) \\ & + \alpha^{\iota_0}_{\vec{m}}(-n_0+1/2, n_0+3/2; -n_0-3/2, n_0-1/2)\end{aligned} \tag{4.49}$$

when $\epsilon = \varepsilon = 0$,

$$\omega^{*}_{1,0} = \alpha^{\iota_0}_{\vec{m}}(1/2, 1/2; -1/2, -1/2), \tag{4.50}$$

$$\omega^{\dagger}_{0,-n_0} = \alpha^{\iota_0}_{\vec{m}}(-n_0+1/2, n_0+1/2; -n_0-1/2, n_0-1/2) \tag{4.51}$$

and

$$\omega^{\dagger}_{1,0} = \alpha^{\iota_0}_{\vec{m}}(3/2, 1/2; -1/2, -3/2) + \alpha^{\iota_0}_{\vec{m}}(1/2, 3/2; -3/2, -1/2). \tag{4.52}$$

Set

$$T^{\tau}_{\epsilon,l} = \vartheta^{\epsilon}_{l+1} - \vartheta^{\epsilon}_l + \omega^{\tau}_{\epsilon,l}\kappa_0 \tag{4.53}$$

for $l \in \mathbb{N} + \delta_{\epsilon,0}(\delta_{\varepsilon,0} - n_0) + \delta_{\epsilon,1}$ if $\tau = *$, and $l \in \mathbb{N} + 1 - n_0\delta_{\epsilon,0}$ when $\tau = \dagger$. Moreover, we set

$$T^{*}_{0,-n_0} = \vartheta^{0}_{-n_0} + \vartheta^{0}_{-n_0+2} + \omega^{*}_{0,-n_0}\kappa_0 \tag{4.54}$$

when $\epsilon = \varepsilon = 0$,

$$T^{*}_{1,0} = \vartheta^{1}_1 + \omega^{*}_{1,0}\kappa_0, \qquad T^{\dagger}_{0,-n_0} = \vartheta^{0}_{-n_0+1} + \omega^{\dagger}_{0,-n_0}\kappa_0 \tag{4.55}$$

and

$$T^{\dagger}_{1,0} = \vartheta^{1}_1 + \vartheta^{1}_2 + \omega^{\dagger}_{1,0}\kappa_0. \tag{4.56}$$

It can be verified that

$$[e^{\tau}_{\epsilon,l}, f^{\tau}_{\epsilon,l}] = T^{\tau}_{\epsilon,l} \quad \text{for } l \in \mathbb{N} - \delta_{0,\epsilon} n_0. \tag{4.57}$$

Obviously

$$[T^{\tau}_{\epsilon,l}, e^{\tau}_{\epsilon,l}] = 2e^{\tau}_{\epsilon,l}, \qquad [T^{\tau}_{\epsilon,l}, f^{\tau}_{\epsilon,l}] = -2f^{\tau}_{\epsilon,l} \quad \text{for } l \in \mathbb{N} - \delta_{0,\epsilon} n_0. \tag{4.58}$$

Set

$$\mathcal{T}^{\epsilon} = \sum_{l=-n_0\delta_{\epsilon,0}}^{\infty} \mathbb{C}\vartheta_l^{\epsilon} + \mathbb{C}\kappa_0. \tag{4.59}$$

Moreover, we let

$$\mathcal{L}_{\iota_0,\epsilon,\pm}^{*,\vec{m}} = \sum_{i,j=1}^{n} \sum_{l,k\in\mathbb{Z};\pm(l+k)>0} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} - \langle l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) \tag{4.60}$$

and

$$\mathcal{L}_{\iota_0,\epsilon,\pm}^{\dagger,\vec{m}} = \sum_{i,j=1}^{n} \sum_{l,k\in\mathbb{Z};\pm(l+k)>0} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} - (-1)^{p(i)+p(j)} \times \langle l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}). \tag{4.61}$$

Then $\mathcal{L}_{\iota_0,\epsilon,\pm}^{\tau,\vec{m}}$ are Lie subalgebras of $\mathcal{L}_{\iota_0,\epsilon}^{\tau,\vec{m}}$, $\mathcal{T}^{\epsilon}$ is a toral Cartan subalgebra of $\mathcal{L}_{\iota_0,\epsilon}^{\tau,\vec{m}}$ and

$$\mathcal{L}_{\iota_0,\epsilon}^{\tau,\vec{m}} = \mathcal{L}_{\iota_0,\epsilon,-}^{\tau,\vec{m}} \oplus \mathcal{T}^{\epsilon} \oplus \mathcal{L}_{\iota_0,\epsilon,+}^{\tau,\vec{m}}. \tag{4.62}$$

Furthermore, we have Borel subalgebras:

$$\mathcal{L}_{\iota_0,\epsilon,0}^{\tau,\vec{m}} = \mathcal{T}^{\epsilon} + \mathcal{L}_{\iota_0,\epsilon,+}^{\tau,\vec{m}} \tag{4.63}$$

of $\mathcal{L}_{\iota_0,\epsilon}^{\tau,\vec{m}}$.

Denote by $(\mathcal{T}^{\epsilon})^*$ the space of linear functions on $\mathcal{T}^{\epsilon}$. Fix an element

$$k_0 \in \mathbb{N} - n_0\delta_{0,\epsilon}. \tag{4.64}$$

Take $\lambda \in (\mathcal{T}^{\epsilon})^*$ such that

$$\lambda(\vartheta_l^{\epsilon}) = 0 \quad \text{for } k_0 \le l \in \mathbb{N} - n_0\delta_{\epsilon,0}. \tag{4.65}$$

Define a one-dimensional $\mathcal{L}_{\iota_0,\epsilon,0}^{\tau,\vec{m}}$-module $\mathbb{C}v_{\lambda}^{\tau,\epsilon}$ by

$$\mathcal{L}_{\iota_0,\epsilon,+}^{\tau,\vec{m}}(v_{\lambda}^{\tau,\epsilon}) = \{0\}, \qquad h(v_{\lambda}^{\tau,\epsilon}) = \lambda(h)v_{\lambda}^{\tau,\epsilon} \quad \text{for } h \in \mathcal{T}^{\epsilon}. \tag{4.66}$$

Form an induced $\mathcal{L}_{\iota,\epsilon}^{\tau,\vec{m}}$-module:

$$M_{\lambda}^{\tau,\epsilon} = U(\mathcal{L}_{\iota_0,\epsilon}^{\tau,\vec{m}}) \otimes_{U(\mathcal{L}_{\iota_0,\epsilon,0}^{\tau,\vec{m}})} \mathbb{C}v_{\lambda}^{\tau,\epsilon} \cong U(\mathcal{L}_{\iota_0,\epsilon,-}^{\tau,\vec{m}}) \otimes_{\mathbb{C}} \mathbb{C}v_{\lambda}^{\tau,\epsilon}. \tag{4.67}$$

There exists a unique maximal proper submodule $N_{\lambda}^{\tau,\epsilon}$ of $M_{\lambda}^{\tau,\epsilon}$, and the quotient module

$$\mathcal{M}_{\lambda}^{\tau,\epsilon} = M_{\lambda}^{\tau,\epsilon}/N_{\lambda}^{\tau,\epsilon} \tag{4.68}$$

is a weighted irreducible $\mathcal{L}_{\iota_0,\epsilon}^{\tau,\vec{m}}$-module. Identify $1 \otimes v_{\lambda}^{\tau,\epsilon}$ with $v_{\lambda}^{\tau,\epsilon}$. If

$$\lambda_l^{\epsilon} = \lambda(T_{\epsilon,l}^{\tau}) \in \mathbb{N} \quad \text{for } l \in \mathbb{N} - n_0\delta_{\epsilon,0}, \tag{4.69}$$

then

$$N_{\lambda}^{\tau,\epsilon} = \sum_{l=-n_0\delta_{0,\epsilon}}^{\infty} U(\mathcal{L}_{\iota_0,\epsilon,-}^{\tau,\vec{m}})(f_{\epsilon,l}^{\tau})^{\lambda_l^{\epsilon}+1}v_{\lambda}^{\tau,\epsilon}. \tag{4.70}$$

Set

$$\hat{\ell} = |\iota_0| + 1 + \max\{m_1, m_2, \ldots, m_n\}. \tag{4.71}$$

For $\hat{\ell} < s \in \mathbb{N}$, we define

$$\mathcal{L}^{*,\vec{m},s}_{\iota_0,\epsilon} = \sum_{i,j=1}^{n} \sum_{l,-k\in\pm(\mathbb{N}+s)} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} \\ -\langle l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) \tag{4.72}$$

and

$$\mathcal{L}^{\dagger,\vec{m},s}_{\iota_0,\epsilon} = \sum_{i,j=1}^{n} \sum_{l,-k\in\pm(\mathbb{N}+s)} \mathbb{C}((-1)^{\epsilon}\langle k+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} \\ -(-1)^{p(i)+p(j)}\langle l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}). \tag{4.73}$$

Then $\mathcal{L}^{\tau,\vec{m},s}_{\iota_0,\epsilon}$ is a Lie subalgebra of $\mathcal{L}^{\tau,\vec{m}}_{\iota,\epsilon}$ by (4.16)–(4.26).

Suppose that $\mathcal{M}$ is an $\mathcal{L}^{\tau,\vec{m}}_{\iota_0,\epsilon}$-module

$$\text{generated by a subspace } \mathcal{M}_0 \text{ such that } \mathcal{L}^{\tau,\vec{m},s}_{\iota_0,\epsilon}(\mathcal{M}_0) = \{0\} \quad \text{for some } \hat{\ell} < s \in \mathbb{N}. \tag{4.74}$$

For instance, the above module $\mathcal{M}^{\tau,\epsilon}_{\lambda}$ is such a module with $\mathcal{M}_0 = \mathbb{C}v^{\tau,\epsilon}_{\lambda}$ and $s = \max\{\hat{\ell}, k_0+2\}$. We can also construct $\mathcal{M}$ as in Example 3.2. According to (2.60), (2.64) and (2.73), we define a representation σ_* of $\hat{o}(\vec{\ell}, \mathbb{A})$ and a representation $\sigma_\dagger$ of $\widehat{sp}(\vec{\ell}, \mathbb{A})$ on $\mathcal{M}$ as follows: $\sigma_*(\kappa) = \kappa_0$, $\sigma_\dagger(\kappa) = \kappa_0$,

$$\sigma_*(t^{k+m_i+m_j+r+\epsilon}\partial_t^{r+\ell_j}E_{i,j} - (-\partial_t)^r t^{k+m_i+m_j+r+\epsilon}\partial_t^{\ell_i}E_{j^*,i^*}) \\ = \sum_{l\in\mathbb{Z}}\langle l-m_j-\epsilon+1/2\rangle_r((-1)^{\epsilon}\langle -l+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{(k+l)n+i-1/2,-ln-j+1/2} \\ -\langle k+l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(-l+\epsilon-1)n-j+1/2,(k+l+1-\epsilon)n+i-1/2}) \\ +((r+\ell_i)!\Im_{0,r+\ell_i} - r!\ell_i!\Im_{r,\ell_i})\delta_{k+\epsilon,0}\delta_{i,j}\kappa_0 \tag{4.75}$$

and

$$\sigma_\dagger(t^{k+m_i+m_j+r+\epsilon}\partial_t^{r+\ell_j}E_{i,j} - (-1)^{p(i)+p(j)}(-\partial_t)^r t^{k+m_i+m_j+r+\epsilon}\partial_t^{\ell_i}E_{j^*,i^*}) \\ = \sum_{l\in\mathbb{Z}}\langle l-m_j-\epsilon+1/2\rangle_r((-1)^{\epsilon}\langle -l+m_j+\epsilon-3/2\rangle_{\ell_j}\mathcal{E}_{(k+l)n+i-1/2,-ln-j+1/2} \\ -(-1)^{p(i)+p(j)}\langle k+l+m_i+1/2\rangle_{\ell_i}\mathcal{E}_{(-l+\epsilon-1)n-j+1/2,(k+l+1-\epsilon)n+i-1/2}) \\ +((r+\ell_i)!\Im_{0,r+\ell_i} - r!\ell_i!\Im_{r,\ell_i})\delta_{k+\epsilon,0}\delta_{i,j}\kappa_0. \tag{4.76}$$

Theorem 4.2. *Suppose that $\mathcal{M}$ is a weighted $\mathcal{L}^{\tau,\vec{m}}_{\iota_0,\epsilon}$-module satisfying (4.74). Then the representation σ_τ is irreducible if and only if $\mathcal{M}$ is irreducible.*

Proof. We only prove the statement for σ_* when $\{\ell_s = 2m_s \mid s \in \overline{1,n}\}$ and $n = 2n_0$ are even. The other cases can be proved similarly. Define

$$h_{i,r} = \sigma_*(t^{\ell_i+r}\partial_t^{r+\ell_i}E_{i,i} - (-\partial_t)^r t^{\ell_i+r}\partial_t^{\ell_i}E_{i^*,i^*}) \\ = \sum_{l\in\mathbb{Z}}\langle l+m_i+1/2\rangle_{2m_i+r}(\mathcal{E}_{ln+i-1/2,-ln-i+1/2} - \mathcal{E}_{-(l+1)n-i+1/2,(l+1)n+i-1/2}) \\ +((r+\ell_i)!\Im_{0,r+\ell_i} - r!\ell_i!\Im_{r,\ell_i})\kappa_0 \tag{4.77}$$

for $i \in \overline{1,n}$ and $r \in \mathbb{N}$ by (4.75). Set

$$H = \sum_{i=1}^{n} \sum_{r=0}^{\infty} \mathbb{C} h_{i,r} \subset \text{End}\,\mathcal{M}, \tag{4.78}$$

the space of linear transformations on $\mathcal{M}$. As operators on $\mathcal{M}$,

$$\begin{aligned}
&[h_{i,r}, \langle k+m_{j_2}-3/2\rangle_{\ell_{j_2}} \mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2} - \langle l+m_{j_1}+1/2\rangle_{\ell_{j_1}} \mathcal{E}_{(k-1)n-j_2+1/2,(l+1)n+j_1-1/2}] \\
&\quad = [\delta_{i,j_1}\langle l+m_i+1/2\rangle_{\ell_i+r} - \delta_{i^*,j_1}\langle -l+m_i-3/2\rangle_{\ell_i+r} + \delta_{i^*,j_2}\langle k+m_i-3/2\rangle_{\ell_i+r} \\
&\qquad - \delta_{i,j_2}\langle -k+m_i+1/2\rangle_{\ell_i+r}](\langle k+m_{j_2}-3/2\rangle_{\ell_{j_2}} \mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2} \\
&\qquad - \langle l+m_{j_1}+1/2\rangle_{\ell_{j_1}} \mathcal{E}_{(k-1)n-j_2+1/2,(l+1)n+j_1-1/2}).
\end{aligned} \tag{4.79}$$

Using generating functions, we get

$$\begin{aligned}
&\left[\sum_{r=0}^{\infty} h_{i,r}x^r, \langle k+m_{j_2}-3/2\rangle_{\ell_{j_2}} \mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2} - \langle l+m_{j_1}+1/2\rangle_{\ell_{j_1}} \mathcal{E}_{(k-1)n-j_2+1/2,(l+1)n+j_1-1/2}\right] \\
&\quad = \frac{\mathrm{d}^{\ell_i}}{\mathrm{d}x^{\ell_i}}[\delta_{i,j_1}(1+x)^{l+m_i+1/2} - \delta_{i^*,j_1}(1+x)^{-l+m_i-3/2} + \delta_{i^*,j_2}(1+x)^{k+m_i-3/2} \\
&\qquad - \delta_{i,j_2}(1+x)^{-k+m_i+1/2}](\langle k+m_{j_2}-3/2\rangle_{\ell_{j_2}} \mathcal{E}_{ln+j_1-1/2,kn-j_2+1/2} \\
&\qquad - \langle l+m_{j_1}+1/2\rangle_{\ell_{j_1}} \mathcal{E}_{(k-1)n-j_2+1/2,(l+1)n+j_1-1/2}).
\end{aligned} \tag{4.80}$$

Denote by H^* the space of linear functions on H. Given $\rho \in H^*$, we set

$$(\mathcal{L}^{*,\vec{m}}_{\iota_0,0})_{(\rho)} = \{\xi \in \mathcal{L}^{*,\vec{m}}_{\iota_0,0} \mid [h,\xi] = \rho(h)\xi \text{ for } h \in H\}. \tag{4.81}$$

Then

$$(\mathcal{L}^{*,\vec{m}}_{\iota_0,0})_{(0)} = \mathcal{T}, \qquad \dim(\mathcal{L}^{*,\vec{m}}_{\iota_0,0})_{(\rho)} = 1 \quad \text{for } 0 \neq \rho \in H^* \tag{4.82}$$

by (4.80). Moreover,

$$\mathcal{L}^{*,\vec{m}}_{\iota_0,0} = \bigoplus_{\rho \in H^*} (\mathcal{L}^{*,\vec{m}}_{1,0})_{(\rho)}. \tag{4.83}$$

The conclusion can be proved exactly as was done for the proof of Theorem 3.1. □

5. Modules with $\iota \in \mathbb{Z}$ related to skew elements

In this section, we give detailed constructions of irreducible modules of the Lie algebras $\hat{\mathcal{A}}^{\tau}_{\vec{\ell}}$ in (2.59) with $\iota \in \mathbb{Z}$ when $\mathcal{A}$ is the $n \times n$ matrix algebra, from weighted irreducible modules of central extensions of the Lie algebras of infinite skew matrices with finite number of nonzero entries.

Recall the Lie algebra $\check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell}}$ defined in (2.76), and the Lie algebra $\tilde{gl}(\infty)$ defined in (3.10) and (3.11). Note that

$$\begin{aligned}
\check{\mathcal{D}}^{*,\iota}_{\vec{\ell}} = \sum_{i,j=1}^{n} \sum_{0<l,k \in \mathcal{Z}} &[\mathbb{C}((-1)^{\epsilon}\langle -k-1/2\rangle_{\ell_j} E_{i,j}(-l-\iota,-k+\iota-\ell_j) - \langle -l-1/2\rangle_{\ell_i} \\
&\times E_{j^*,i^*}(-k-\iota,-l+\iota-\ell_i)) + \mathbb{C}((-1)^{\epsilon}\langle -k-1/2\rangle_{\ell_j} E_{i,j}(l-\iota+\ell_i,-k+\iota-\ell_j) \\
&- \langle l+\ell_i-1/2\rangle_{\ell_i} E_{j^*,i^*}(-k-\iota,l+\iota)) + \mathbb{C}((-1)^{\epsilon}\langle k+\ell_j-1/2\rangle_{\ell_j} E_{i,j}(l-\iota+\ell_i,k+\iota) \\
&- \langle l+\ell_i-1/2\rangle_{\ell_i} E_{j^*,i^*}(k-\iota+\ell_j,l+\iota))] + \mathbb{C}\kappa_0
\end{aligned} \tag{5.1}$$

and

$$\check{\mathcal{D}}^{\dagger,\iota}_{\vec{\ell}} = \sum_{i,j=1}^{n} \sum_{0<l,k \in \mathcal{Z}} [\mathbb{C}((-1)^{\epsilon}\langle -k-1/2\rangle_{\ell_j} E_{i,j}(-l-\iota,-k+\iota-\ell_j)$$

$$
\begin{aligned}
&-(-1)^{p(i)+p(j)}\langle -l-1/2\rangle_{\ell_i} E_{j^*,i^*}(-k-\iota,-l+\iota-\ell_i))\\
&+\mathbb{C}((-1)^{\epsilon}\langle -k-1/2\rangle_{\ell_j} E_{i,j}(l-\iota+\ell_i,-k+\iota-\ell_j)\\
&-(-1)^{p(i)+p(j)}\langle l+\ell_i-1/2\rangle_{\ell_i} E_{j^*,i^*}(-k-\iota,l+\iota))\\
&+\mathbb{C}((-1)^{\epsilon}\langle k+\ell_j-1/2\rangle_{\ell_j} E_{i,j}(l-\iota+\ell_i,k+\iota)\\
&-(-1)^{p(i)+p(j)}\langle l+\ell_i-1/2\rangle_{\ell_i} E_{j^*,i^*}(k-\iota+\ell_j,l+\iota))]+\mathbb{C}\kappa_0
\end{aligned}
\tag{5.2}
$$

if n is even. By (3.12), we set

$$
\begin{aligned}
\mathcal{G}_{\vec{\ell}}^{*,\iota} = \sum_{i,j=1}^{n}\sum_{l,k=0}^{\infty}&[\mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{-(l+\iota+1)n+i-1/2,(-k+\iota-\ell_j)n-j+1/2}\\
&-\langle -l-1\rangle_{\ell_i}\mathcal{E}_{-(k+\iota)n-j+1/2,(-l+\iota-\ell_i-1)n+i-1/2})\\
&+\mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(l-\iota+\ell_i)n+i-1/2,(-k+\iota-\ell_j)n-j+1/2}-\langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{-(k+\iota)n-j+1/2,(l+\iota)n+i-1/2})\\
&+\mathbb{C}((-1)^{\epsilon}\langle k+\ell_j\rangle_{\ell_j} E_{(l-\iota+\ell_i)n+i-1/2,(k+\iota+1)n-j+1/2}\\
&-\langle l+\ell_i\rangle_{\ell_i} E_{(k-\iota+\ell_j+1)n-j+1/2,(l+\iota)n+i-1/2})]+\mathbb{C}\kappa_0
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
\mathcal{G}_{\vec{\ell}}^{\dagger,\iota} = \sum_{i,j=1}^{n}\sum_{l,k=0}^{\infty}&[\mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{-(l+\iota+1)n+i-1/2,(-k+\iota-\ell_j)n-j+1/2}\\
&-(-1)^{p(i)+p(j)}\langle -l-1\rangle_{\ell_i}\mathcal{E}_{-(k+\iota)n-j+1/2,(-l+\iota-\ell_i-1)n+i-1/2})\\
&+\mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(l-\iota+\ell_i)n+i-1/2,(-k+\iota-\ell_j)n-j+1/2}\\
&-(-1)^{p(i)+p(j)}\langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{-(k+\iota)n-j+1/2,(l+\iota)n+i-1/2})\\
&+\mathbb{C}((-1)^{\epsilon}\langle k+\ell_j\rangle_{\ell_j} E_{(l-\iota+\ell_i)n+i-1/2,(k+\iota+1)n-j+1/2}\\
&-(-1)^{p(i)+p(j)}\langle l+\ell_i\rangle_{\ell_i} E_{(k-\iota+\ell_j+1)n-j+1/2,(l+\iota)n+i-1/2})]+\mathbb{C}\kappa_0.
\end{aligned}
\tag{5.4}
$$

Then $\mathcal{G}_{\vec{\ell}}^{*,\iota}$ and $\mathcal{G}_{\vec{\ell}}^{\dagger,\iota}$ are Lie subalgebras of $\tilde{gl}(\infty)$, and the map in (3.12) induces

$$
\check{\mathcal{D}}_{\vec{\ell}}^{\tau,\iota}\cong\mathcal{G}_{\vec{\ell}}^{\tau,\iota}
\tag{5.5}
$$

(cf. (4.44)).

We define two functions $\hat{H}_1,\hat{H}_2:\mathcal{Z}\to\mathbb{C}$ by

$$
\hat{H}_1(l)=\begin{cases}1 & \text{if } n\iota<l<0 \text{ or } l>0,(\iota-\ell_{(l+1/2)_R})n,\\ 0 & \text{otherwise}\end{cases}
\tag{5.6}
$$

and

$$
\hat{H}_2(l)=\begin{cases}1 & \text{if } -\iota n,0<l \text{ or } (\ell_{(-l+1/2)_R}-\iota)n<l<0,\\ 0 & \text{otherwise}\end{cases}
\tag{5.7}
$$

(cf. (4.25)). Moreover, we define a map $\beta_{\vec{\ell}}^{\iota}:\mathcal{Z}^4\to\mathbb{C}$ by:

$$
\beta_{\vec{\ell}}^{\iota}(l_1,l_2;k_1,k_2)=(H_1(l_1)H_2(l_2)-H_1(k_1)H_2(k_2))\delta_{l_1+k_2,0}\delta_{l_2+k_1,0}.
\tag{5.8}
$$

Set

$$
\tilde{gl}_{\vec{\ell}}^{(\iota)}(\infty)=\overline{gl}(\infty)\oplus\mathbb{C}\kappa_0
\tag{5.9}
$$

(cf. (3.1)), where κ_0 is a base element. We have the following Lie bracket on $\tilde{gl}_{\vec{\ell}}^{(\iota)}(\infty)$:

$$
[\mathcal{E}_{l_1,l_2}+\mu_1\kappa_0,\mathcal{E}_{k_1,k_2}+\mu_2\kappa_0]=\mathcal{E}_{l_1,l_2}\mathcal{E}_{k_1,k_2}-\mathcal{E}_{k_1,k_2}\mathcal{E}_{l_1,l_2}+\beta_{\vec{\ell}}^{\iota}(l_1,l_2;k_1,k_2)\kappa_0
\tag{5.10}
$$

for $l_1, l_2, k_1, k_2 \in \mathcal{Z}$. In particular, the Lie algebras

$$\tilde{gl}^{(0)}_{\vec{\ell}}(\infty) \cong \tilde{gl}(\infty). \tag{5.11}$$

Next we set

$$\begin{aligned}\mathcal{L}^{*,\vec{\ell}}_{\iota} = \sum_{i,j=1}^{n}\sum_{l,k=0}^{\infty} & [\mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} - \langle -l-1\rangle_{\ell_i} \\ & \times \mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2}) + \mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,-kn-j+1/2} \\ & - \langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,ln+i-1/2}) + \mathbb{C}((-1)^{\epsilon}\langle k+\ell_j\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2} \\ & - \langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2})] + \mathbb{C}\kappa_0\end{aligned} \tag{5.12}$$

and

$$\begin{aligned}\mathcal{L}^{\dagger,\vec{\ell}}_{\iota} = \sum_{i,j=1}^{n}\sum_{l,k=0}^{\infty} & [\mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} - (-1)^{p(i)+p(j)}\langle -l-1\rangle_{\ell_i} \\ & \times \mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2}) + \mathbb{C}((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,-kn-j+1/2} - (-1)^{p(i)+p(j)} \\ & \times \langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,ln+i-1/2}) + \mathbb{C}((-1)^{\epsilon}\langle k+\ell_j\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2} \\ & - (-1)^{p(i)+p(j)}\langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2})] + \mathbb{C}\kappa_0\end{aligned} \tag{5.13}$$

if n is even. Then $\mathcal{L}^{\tau,\vec{\ell}}_{\iota}$ are Lie subalgebras of $\tilde{gl}^{(\iota)}_{\vec{\ell}}(\infty)$, and

$$\check{\mathcal{D}}^{\tau,\iota}_{\vec{\ell}} \cong \mathcal{G}^{\tau,\iota}_{\vec{\ell}} \cong \mathcal{L}^{\tau,\vec{\ell}}_{\iota} \tag{5.14}$$

(cf. (4.44)) by (5.3) and (5.4). Thus we have:

Theorem 5.1. *The Lie algebras:*

$$\check{\mathcal{D}}^{*,\iota}_{\vec{\ell}}/\mathbb{C}\kappa_0 \cong \begin{cases}\bar{o}_d(\infty) & \text{if } \epsilon = 0,\\ \overline{sp}(\infty) & \text{if } \epsilon = 1\end{cases} \tag{5.15}$$

and

$$\check{\mathcal{D}}^{\dagger,0}_{\vec{\ell}}/\mathbb{C}\kappa_0 \cong \begin{cases}\overline{sp}(\infty) & \text{if } \epsilon = 0,\\ \bar{o}_d(\infty) & \text{if } \epsilon = 1\end{cases} \tag{5.16}$$

(cf. (4.1) *and* (4.3)*).*

Next we want to study the highest-weight irreducible modules of $\mathcal{L}^{\tau,\vec{\ell}}_{\iota}$. Recall the notions in (4.25). Set

$$\begin{aligned}\mathcal{L}^{*,\vec{\ell}}_{\iota,+} &= \sum_{0<k<l}\mathbb{C}((-1)^{\epsilon}\langle -k_Q-1\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,-k+1/2} - \langle l_Q+\ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,l-1/2}) \\ &\quad + \sum_{l,k=1}^{\infty}\mathbb{C}((-1)^{\epsilon}\langle k_Q+\ell_{k_R}\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,k-1/2} - \langle l_Q+\ell_R\rangle_{\ell_{l_R}}\mathcal{E}_{k-1/2,l-1/2}),\end{aligned} \tag{5.17}$$

$$\begin{aligned}\mathcal{L}^{*,\vec{\ell}}_{\iota,-} &= \sum_{l,k=1}^{\infty}\mathbb{C}((-1)^{\epsilon}\langle -k_Q-1\rangle_{\ell_{k_R}}\mathcal{E}_{-l+1/2,-k+1/2} - \langle -l_Q-1\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,-l+1/2}) \\ &\quad + \sum_{0<l<k}\mathbb{C}((-1)^{\epsilon}\langle -k_Q-1\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,-k+1/2} - \langle l_Q+\ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,l-1/2})\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}\mathcal{L}_{\iota,+}^{\dagger,\vec{\ell}} = & \sum_{0<k<l} \mathbb{C}((-1)^{\epsilon}\langle -k_Q - 1\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,-k+1/2} - (-1)^{p(l_R)+p(k_R)} \\ & \times \langle l_Q + \ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,l-1/2}) + \sum_{l,k=1}^{\infty} \mathbb{C}((-1)^{\epsilon}\langle k_Q + \ell_{k_R}\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,k-1/2} \\ & - (-1)^{p(l_R)+p((k_R)^*)}\langle l_Q + \ell_R\rangle_{\ell_{l_R}}\mathcal{E}_{k-1/2,l-1/2}),\end{aligned} \tag{5.19}$$

$$\begin{aligned}\mathcal{L}_{\iota,-}^{\dagger,\vec{\ell}} = & \sum_{l,k=1}^{\infty} \mathbb{C}((-1)^{\epsilon}\langle -k_Q - 1\rangle_{\ell_{k_R}}\mathcal{E}_{-l+1/2,-k+1/2} - (-1)^{p((l_R)^*)+p(k_Q)} \\ & \times \langle -l_Q - 1\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,-l+1/2}) + \sum_{0<l<k} \mathbb{C}((-1)^{\epsilon}\langle -k_Q - 1\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,-k+1/2} \\ & - (-1)^{p(l_R)+p(k_R)}\langle l_Q + \ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,l-1/2}).\end{aligned} \tag{5.20}$$

Then $\mathcal{L}_{\iota,\pm}^{\tau,\vec{\ell}}$ are Lie subalgebras of $\mathcal{L}_{\iota}^{\tau,\vec{\ell}}$ (cf. (4.44)).

Denote

$$\vartheta_l = \mathcal{E}_{l+1/2,-l-1/2} - \mathcal{E}_{-l-1/2,l+1/2} \quad \text{for } l \in \mathbb{N}. \tag{5.21}$$

Set

$$\mathcal{T} = \sum_{l=0}^{\infty} \vartheta_l + \mathbb{C}\kappa_0. \tag{5.22}$$

Then $\mathcal{T}$ is a toral Cartan subalgebra of $\mathcal{L}_{\iota}^{\tau,\vec{\ell}}$. Moreover,

$$\mathcal{L}_{\iota}^{\tau,\vec{\ell}} = \mathcal{L}_{\iota,-}^{\tau,\vec{\ell}} \oplus \mathcal{T} \oplus \mathcal{L}_{\iota,+}^{\tau,\vec{\ell}}. \tag{5.23}$$

Define

$$f_{\epsilon,l}^{*} = \langle -(l+1)_Q - 1\rangle_{\ell_{(l+1)_R}}\mathcal{E}_{l-1/2,-l-1/2} - (-1)^{\epsilon}\langle l_Q + \ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-l-1/2,l-1/2} \tag{5.24}$$

and

$$f_{\epsilon,l}^{\dagger} = \langle -(l+1)_Q - 1\rangle_{\ell_{(l+1)_R}}\mathcal{E}_{l-1/2,-l-1/2} - (-1)^{\epsilon+p(l_R)+p((l+1)_R)}\langle l_Q + \ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-l-1/2,l-1/2} \tag{5.25}$$

for $l \in \mathbb{N}+1$. Moreover, we define

$$f_{0,0}^{*} = \langle -1\rangle_{\ell_1}\mathcal{E}_{-3/2,-1/2} - \langle -2_Q - 1\rangle_{\ell_{2_R}}\mathcal{E}_{-1/2,-3/2}, \tag{5.26}$$

$$f_{1,0}^{\dagger} = \langle -1\rangle_{\ell_1}\mathcal{E}_{-3/2,-1/2} + (-1)^{p(1)+p((2_R)^*)}\langle -2_Q - 1\rangle_{\ell_{2_R}}\mathcal{E}_{-1/2,-3/2}, \tag{5.27}$$

$$f_{1,0}^{*} = f_{0,0}^{\dagger} = \mathcal{E}_{-1/2,-1/2}. \tag{5.28}$$

Then $\{f_{\epsilon,l}^{\tau} \mid l \in \mathbb{N}\}$ is a set of negative simple root vectors of $\mathcal{L}_{\iota}^{\tau,\vec{\ell}}$.

For $l \in \mathbb{N}+1$, we define

$$\omega_l = \beta_{\vec{\ell}}^{\iota}(l+1/2, -l+1/2; l-1/2, -l-1/2) + \beta_{\vec{\ell}}^{\iota}(-l+1/2, l+1/2; -l-1/2, l-1/2). \tag{5.29}$$

Moreover, we let

$$\omega_{0,0}^{*} = \beta_{\vec{\ell}}^{\iota}(1/2, 3/2; -3/2, -1/2) + \beta_{\vec{\ell}}^{\iota}(3/2, 1/2; -1/2, -3/2) \tag{5.30}$$

and

$$\omega_{1,0}^{*} = \beta_{\vec{\ell}}^{\iota}(1/2, 1/2; -1/2, -1/2). \tag{5.31}$$

Furthermore, we set

$$T^{\tau}_{\epsilon,l} = \vartheta_l - \vartheta_{l-1} + \omega_l \kappa_0 \quad \text{for } l \in \mathbb{N}+1, \tag{5.32}$$

$$T^*_{0,0} = T^{\dagger}_{1,0} = \vartheta_1 + \vartheta_0 + \omega^*_{0,0}\kappa_0 \tag{5.33}$$

and

$$T^*_{1,0} = T^{\dagger}_{0,0} = \vartheta_0 + \omega^*_{1,0}\kappa_0. \tag{5.34}$$

Denote

$$\mathcal{L}^{\tau,\vec{\ell}}_{\iota,0} = \mathcal{T} + \mathcal{L}^{\tau,\vec{\ell}}_{\iota,+}. \tag{5.35}$$

Let $\lambda^{\tau,\epsilon}$ be a linear function on $\mathcal{T}$ such that there exists $k_0 \in \mathbb{N}$ for which

$$\lambda^{\tau,\epsilon}(\vartheta_l) = 0 \quad \text{for } k_0 \leq l \in \mathbb{N}. \tag{5.36}$$

Define a one-dimensional $\mathcal{L}^{\tau,\vec{\ell}}_{\iota,0}$-module $\mathbb{C}v^{\tau,\epsilon}$ by:

$$\mathcal{L}^{\tau,\vec{\ell}}_{\iota,+}(v^{\tau,\epsilon}) = \{0\}, \qquad h(v^{\tau,\epsilon}) = \lambda^{\tau,\epsilon}(h)v^{\tau,\epsilon} \quad \text{for } h \in \mathcal{T}. \tag{5.37}$$

Form an induced $\mathcal{L}^{\tau,\vec{\ell}}_{\iota}$-module:

$$M_{\lambda^{\tau,\epsilon}} = U(\mathcal{L}^{\tau,\vec{\ell}}_{\iota}) \otimes_{U(\mathcal{L}^{\tau,\vec{\ell}}_{\iota,0})} \mathbb{C}v^{\tau,\epsilon} \cong U(\mathcal{L}^{\tau,\vec{\ell}}_{\iota,-}) \otimes_{\mathbb{C}} \mathbb{C}v^{\tau,\epsilon}. \tag{5.38}$$

There exists a unique maximal proper submodule $N_{\lambda^{\tau,\epsilon}}$ of $M_{\lambda^{\tau,\epsilon}}$, and the quotient

$$\mathcal{M}_{\lambda^{\tau,\epsilon}} = M_{\lambda^{\tau,\epsilon}}/N_{\lambda^{\tau,\epsilon}} \tag{5.39}$$

is a weighted irreducible $\mathcal{L}^{\tau,\vec{\ell}}_{\iota}$-module. Identify $1 \otimes v^{\tau,\epsilon}$ with $v^{\tau,\epsilon}$. When

$$\lambda^{\tau,\epsilon}_k = \lambda^{\tau,\epsilon}(T^{\tau}_{\epsilon,k}) \in \mathbb{N} \quad \text{for } k \in \mathbb{N}, \tag{5.40}$$

the submodule

$$N_{\lambda^{\tau,\epsilon}} = \sum_{l=0}^{\infty} U(\mathcal{L}^{\tau,\vec{\ell}}_{\iota,-})(f^{\tau}_{\epsilon,l})^{\lambda^{\tau,\epsilon}_l+1}v^{\tau,\epsilon}. \tag{5.41}$$

Based on (5.3) and (5.4), we denote

$$\bar{\ell} = (|\iota| + \max\{\ell_1, \ell_2, \ldots, \ell_n\})n. \tag{5.42}$$

For any $\bar{\ell} < s \in \mathbb{N}$, we define:

$$\mathcal{L}^{*,\vec{\ell}}_{\iota,s} = \sum_{l,k=s}^{\infty} \mathbb{C}((-1)^{\epsilon}\langle -k_Q - 1\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,-k+1/2} - \langle l_Q + \ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,l-1/2}) \tag{5.43}$$

and

$$\mathcal{L}^{\dagger,\vec{\ell}}_{\iota,s} = \sum_{l,k=s}^{\infty} \mathbb{C}((-1)^{\epsilon}\langle -k_Q - 1\rangle_{\ell_{k_R}}\mathcal{E}_{l-1/2,-k+1/2} - (-1)^{p(l_R)+p(k_R)}\langle l_Q + \ell_{l_R}\rangle_{\ell_{l_R}}\mathcal{E}_{-k+1/2,l-1/2}). \tag{5.44}$$

Suppose that $\mathcal{M}$ is an $\mathcal{L}^{\tau,\vec{\ell}}_{\iota}$-module

$$\text{generated by a subspace } \mathcal{M}_0 \text{ such that } \mathcal{L}^{\tau,\vec{\ell}}_{\iota,s}(\mathcal{M}_0) = \{0\} \text{ for some } \bar{\ell} < s \in \mathbb{N}. \tag{5.45}$$

For instance, the above module $\mathcal{M}^{\tau,\epsilon}$ is such a module with $\mathcal{M}_0 = \mathbb{C}v^{\tau,\epsilon}$ and $s = \max\{k_0 + 2, \bar{\ell} + 1\}$. We can also construct $\mathcal{M}$ as in Example 3.2.

Recall the notion defined in (2.60). For $i, j \in \overline{1, n}$ and $r \in \mathbb{N}$, we have

$$(E_{i,j})^*_{\vec{\ell}}(r, z) = \sum_{l \in \mathbb{Z}} (E_{i,j} \otimes t^l \partial_t^{r+\ell_j} - (-1)^{\epsilon} E_{j^*,i^*} \otimes (-\partial_t)^r t^l \partial_t^{\ell_i}) z^{-l-1} \tag{5.46}$$

and

$$(E_{i,j})^{\dagger}_{\vec{\ell}}(r, z) = \sum_{l \in \mathbb{Z}} (E_{i,j} \otimes t^l \partial_t^{r+\ell_j} - (-1)^{\epsilon+p(i)+p(j)} E_{j^*,i^*} \otimes (-\partial_t)^r t^l \partial_t^{\ell_i}) z^{-l-1}. \tag{5.47}$$

By Theorem 2.5, we have a representation σ_* of $\hat{o}(\vec{\ell}, \mathbb{A})$ and a representation $\sigma_\dagger$ of $\widehat{sp}(\vec{\ell}, \mathbb{A})$ on $\mathcal{M}$ with $\sigma_*(\kappa) = \kappa_0, \sigma_\dagger(\kappa) = \kappa_0$ and

$$\begin{aligned}
\sigma_*((E_{i,j})^*_{\vec{\ell}}(r, z)) = \sum_{l,k=0}^{\infty} [& \langle k \rangle_r ((-1)^{\epsilon} \langle -k-1 \rangle_{\ell_j} \mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} \\
& - \langle -l-1 \rangle_{\ell_i} \mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2}) z^{l+k-r} \\
& + \langle k \rangle_r ((-1)^{\epsilon} \langle -k-1 \rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,-kn-j+1/2} \\
& - \langle l+\ell_i \rangle_{\ell_i} \mathcal{E}_{-kn-j+1/2,ln+i-1/2}) z^{-l+k-\ell_i-r-1} \\
& + \langle -k-\ell_j-1 \rangle_r ((-1)^{\epsilon} \langle k+\ell_j \rangle_{\ell_j} \mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2} \\
& - \langle -l-1 \rangle_{\ell_i} \mathcal{E}_{(k+1)n-j+1/2,-(l+1)n+i-1/2}) z^{l-k-\ell_j-r-1} \\
& + \langle -k-\ell_j-1 \rangle_r ((-1)^{\epsilon} \langle k+\ell_j \rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2} \\
& - \langle l+\ell_i \rangle_{\ell_i} \mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2}) z^{-l-k-\ell_i-\ell_j-r-2} \\
& + [(r+\ell_i)! \Im_{0,r+\ell_i} - (-1)^{\epsilon} r! \ell_i! \Im_{r,\ell_i}] \delta_{i,j} \kappa_0 z^{-r-\ell_i-1}]
\end{aligned} \tag{5.48}$$

and

$$\begin{aligned}
\sigma_\dagger((E_{i,j})^{\dagger}_{\vec{\ell}}(r, z)) = \sum_{l,k=0}^{\infty} [& \langle k \rangle_r ((-1)^{\epsilon} \langle -k-1 \rangle_{\ell_j} \mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} \\
& - (-1)^{p(i)+p(j)} \langle -l-1 \rangle_{\ell_i} \mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2}) z^{l+k-r} \\
& + \langle k \rangle_r ((-1)^{\epsilon} \langle -k-1 \rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,-kn-j+1/2} \\
& - (-1)^{p(i)+p(j)} \langle l+\ell_i \rangle_{\ell_i} \mathcal{E}_{-kn-j+1/2,ln+i-1/2}) z^{-l+k-\ell_i-r-1} \\
& + \langle -k-\ell_j-1 \rangle_r ((-1)^{\epsilon} \langle k+\ell_j \rangle_{\ell_j} \mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2} \\
& - (-1)^{p(i)+p(j)} \langle -l-1 \rangle_{\ell_i} \mathcal{E}_{(k+1)n-j+1/2,-(l+1)n+i-1/2}) z^{l-k-\ell_j-r-1} \\
& + \langle -k-\ell_j-1 \rangle_r ((-1)^{\epsilon} \langle k+\ell_j \rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2} \\
& - (-1)^{p(i)+p(j)} \langle l+\ell_i \rangle_{\ell_i} \mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2}) z^{-l-k-\ell_i-\ell_j-r-2} \\
& + [(r+\ell_i)! \Im_{0,r+\ell_i} - (-1)^{\epsilon} r! \ell_i! \Im_{r,\ell_i}] \delta_{i,j} \kappa_0 z^{-r-\ell_i-1}].
\end{aligned} \tag{5.49}$$

By a similar proof as that of Theorem 4.2, we obtain the following theorem, which was proved by Ma [29] in a different form when $\iota = 0$.

Theorem 5.2. *Suppose that $\mathcal{M}$ is a weighted $\mathcal{L}_{\iota}^{\tau,\vec{\ell}}$-module satisfying (5.45). Then the representation σ_τ is irreducible if and only if $\mathcal{M}$ is irreducible.*

6. Vacuum representation of $\widehat{gl}(\vec{\ell}, \mathbb{A})$

In this section, we study the vacuum representation of the Lie algebra $\widehat{gl}(\vec{\ell}, \mathbb{A})$ in (3.7) and its vertex algebra structure. Its vertex algebra irreducible representations are investigated.

Recall the algebra $\mathbb{A}$ of differential operators given in (2.2) and (2.3). Observe that

$$\mathbb{A}_- = \sum_{i=0}^{\infty} \mathbb{C}[t^{-1}]t^{-1}\partial_t^i \quad \text{and} \quad \mathbb{A}_+ = \sum_{i=0}^{\infty} \mathbb{C}[t]\partial_t^i \tag{6.1}$$

forms associative subalgebras of $\mathbb{A}$. Moreover,

$$\mathbb{A} = \mathbb{A}_- + \mathbb{A}_+. \tag{6.2}$$

Recall the Lie algebra $\widehat{gl}(n, \mathbb{A})$ in (3.4) and (3.5). Note that in (3.5),

$$\delta_{r_1+r_2,m_1+m_2} r_1! r_2! \binom{m_1}{r_1 + r_2 + 1} = 0 \qquad \text{for } m_1, m_2, r_1, r_2 \in \mathbb{N}, \tag{6.3}$$

because if $m_1 + m_2 = r_1 + r_2$, then $m_1 < r_1 + r_2 + 1$. Thus we have the following Lie subalgebras of $\widehat{gl}(n, \mathbb{A})$:

$$\widehat{gl}(n, \mathbb{A})_\pm = M_{n\times n}(\mathbb{A}_\pm). \tag{6.4}$$

Moreover,

$$\widehat{gl}(n, \mathbb{A}) = \widehat{gl}(n, \mathbb{A})_- + \widehat{gl}(n, \mathbb{A})_+ + \mathbb{C}\kappa. \tag{6.5}$$

Suppose that $\mathcal{G}$ is the Lie subalgebra of $\widehat{gl}(n, \mathbb{A})$ such that

$$\mathcal{G} = \mathcal{G}_- + \mathcal{G}_+ + \mathbb{C}\kappa, \qquad \mathcal{G}_\pm = \mathcal{G} \bigcap \widehat{gl}(n, \mathbb{A})_\pm. \tag{6.6}$$

Then $\mathcal{G}_\pm$ and

$$\mathcal{B} = \mathcal{G}_+ + \mathbb{C}\kappa \tag{6.7}$$

are Lie subalgebras of $\mathcal{G}$. Take a nonzero constant $\chi \in \mathbb{C}$. Form a one-dimensional $\mathcal{B}$-module $\mathbb{C}|0\rangle$ by:

$$\kappa(|0\rangle) = \chi|0\rangle, \qquad \mathcal{G}_+(|0\rangle) = \{0\}. \tag{6.8}$$

The induced $\mathcal{G}$-module

$$\mathcal{V}_\chi(\mathcal{G}) = U(\mathcal{G}) \otimes_{U(\mathcal{B})} \mathbb{C}|0\rangle \cong U(\mathcal{G}_-) \otimes_{\mathbb{C}} \mathbb{C}|0\rangle \tag{6.9}$$

is called the *vacuum module* of $\mathcal{G}$ and the corresponding representation is called the *vacuum representation of* $\mathcal{G}$ *at level* χ. The main objective in the rest of paper is to study $\mathcal{V}_\chi(\mathcal{G})$ and the related vertex algebra structure when $\mathcal{G}$ is one of the Lie algebras $\widehat{gl}(\vec{\ell}, \mathbb{A})$, $\hat{o}(\vec{\ell}, \mathbb{A})$ or $\widehat{sp}(\vec{\ell}, \mathbb{A})$. For convenience, we simply denote

$$u|0\rangle = u \otimes |0\rangle \quad \text{for } u \in U(\mathcal{G}). \tag{6.10}$$

In the rest of this section, we will deal only with $\widehat{gl}(\vec{\ell}, \mathbb{A})$.

Theorem 6.1. *The module $\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$ is irreducible if $\chi \notin \mathbb{Z}$. When $\chi \in \mathbb{Z}$, the module $\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$ has a unique maximal proper submodule $\bar{\mathcal{V}}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$, and the quotient*

$$V_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})) = \mathcal{V}(\widehat{gl}(\vec{\ell}, \mathbb{A}))/\bar{\mathcal{V}}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})) \tag{6.11}$$

is an irreducible $\widehat{gl}(\vec{\ell}, \mathbb{A})$-module. If $n > 1$ and $\chi \in \mathbb{N}$, the submodule

$$\bar{\mathcal{V}}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})) = U(\widehat{gl}(\vec{\ell}, \mathbb{A}))(t^{-1}\partial^{\ell_1} E_{n,1})^{\chi+1}|0\rangle. \tag{6.12}$$

Proof. We define

$$\widehat{gl}(\vec{\ell}, \mathbb{A})_{(k)} = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \mathbb{C}t^{r+\ell_j-k}\partial_t^{r+\ell_j} E_{i,j} + \mathbb{C}\delta_{k,0}\kappa \tag{6.13}$$

for $k \in \mathbb{Z}$. Then

$$\widehat{gl}(\vec{\ell}, \mathbb{A}) = \bigoplus_{k\in\mathbb{Z}} \widehat{gl}(\vec{\ell}, \mathbb{A})_{(k)} \tag{6.14}$$

is a $\mathbb{Z}$-graded Lie algebra. Moreover, we define a $\mathbb{Z}$-grading on $\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$ by

$$\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))_{(0)} = \mathbb{C}|0\rangle, \qquad \mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))_{(-m)} = \{0\} \quad \text{for } m \in \mathbb{N}+1 \tag{6.15}$$

and

$$\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))_{(m)} = \text{Span}\left\{u_1 u_2 \cdots u_s|0\rangle \mid u_i \in \widehat{gl}(\vec{\ell}, \mathbb{A})_- \bigcap \widehat{gl}(\vec{\ell}, \mathbb{A})_{(k_i)}; \sum_{i=1}^{s} k_i = m\right\} \tag{6.16}$$

for $m \in \mathbb{N}+1$. Then

$$\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})) = \bigoplus_{k\in\mathbb{Z}} \mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))_{(k)} \tag{6.17}$$

is a $\mathbb{Z}$-graded $\widehat{gl}(\vec{\ell}, \mathbb{A})$-module. Since

$$(\mathbb{A}\partial_t^{\ell_j} E_{i,j}) \bigcap \widehat{gl}(\vec{\ell}, \mathbb{A})_{(k+\ell_j)} \bigcap \widehat{gl}(\vec{\ell}, \mathbb{A})_- = \sum_{r=0}^{k-1} \mathbb{C}t^{r-k}\partial_t^{r+\ell_j} E_{i,j} \tag{6.18}$$

(cf. (6.4) and (6.6)), the character

$$d(\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})), q) = \sum_{k=0}^{\infty} (\dim \mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))_{(k)}) q^k = \prod_{i=1}^{n}\prod_{r=1}^{\infty} \frac{1}{(1-q^{\ell_i+r})^{rn}}. \tag{6.19}$$

Recall the Lie algebra $\tilde{gl}(\infty)$ defined in (3.10) and (3.11). Set

$$\tilde{gl}(\infty)_{(-)} = \sum_{0>j,k\in\mathcal{Z}} \mathbb{C}\mathcal{E}_{j,k}, \;\; \tilde{gl}(\infty)_{(+)} = \sum_{0<j,k\in\mathcal{Z}} (\mathbb{C}\mathcal{E}_{j,k} + \mathbb{C}\mathcal{E}_{j,-k} + \mathbb{C}\mathcal{E}_{-j,k}). \tag{6.20}$$

By (3.11), $\tilde{gl}(\infty)_{(\pm)}$ are Lie subalgebras of $\tilde{gl}(\infty)$ and

$$\tilde{gl}(\infty) = \tilde{gl}(\infty)_{(-)} + \tilde{gl}(\infty)_{(+)} + \mathbb{C}\kappa_0. \tag{6.21}$$

Hence we have the Lie subalgebra

$$\tilde{gl}(\infty)_{(0)} = \tilde{gl}(\infty)_{(+)} + \mathbb{C}\kappa_0. \tag{6.22}$$

Define a one-dimensional $\tilde{gl}(\infty)_{(0)}$-module $\mathbb{C}\mathbf{1}$ by

$$\kappa_0(\mathbf{1}) = \chi\mathbf{1}, \qquad \tilde{gl}(\infty)_{(+)}(\mathbf{1}) = \{0\}. \tag{6.23}$$

Form an induced $\tilde{gl}(\infty)$-module

$$U_\chi = U(\tilde{gl}(\infty)) \otimes_{U(\tilde{gl}(\infty)_{(-)})} \mathbb{C}\mathbf{1} \cong U(\tilde{gl}(\infty)_{(-)}) \otimes_{\mathbb{C}} \mathbb{C}\mathbf{1}, \tag{6.24}$$

which satisfies the condition (3.19) with $m=0$ and $\mathcal{M}_0 = \mathbb{C}1 \otimes \mathbf{1}$. For convenience, we denote

$$v\mathbf{1} = v \otimes \mathbf{1} \quad \text{for } v \in U(\tilde{gl}(\infty)). \tag{6.25}$$

Note our notion

$$E_{i,j}(r,z) = \sum_{m\in\mathbb{Z}} t^m \partial_t^r z^{-m-1} E_{i,j} \quad \text{for } i,j \in \overline{1,n}, r \in \mathbb{N}+\ell_j. \tag{6.26}$$

According to (3.38) (also cf. (2.52) and (3.12)), we obtain a $\widehat{gl}(\vec{\ell},\mathbb{A})$-module structure on U_χ defined by $\kappa=\chi\mathrm{Id}_{U_\chi}$ and

$$\begin{aligned}E_{i,j}(r,z)=&\sum_{l,k=0}^{\infty}[\langle -k-1\rangle_r(\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2}z^{-l-k-\ell_i-r-2}+\mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2}z^{l-k-r-1})\\&+\langle k+\ell_j\rangle_r(\mathcal{E}_{ln+i-1/2,-kn-j+1/2}z^{-l+k+\ell_j-\ell_i-r-1}+\mathcal{E}_{-(l+1)n+i-1/2,-kn-j+1/2}z^{l+k+\ell_j-r})]\end{aligned}\tag{6.27}$$

for $i,j\in\overline{1,n}$ and $r\in\mathbb{N}+\ell_j$. In particular,

$$E_{i,j}(r,z)(\mathbf{1})=\sum_{l,k=0}^{\infty}\mathcal{E}_{-(l+1)n+i-1/2,-kn-j+1/2}\mathbf{1}z^{l+k+\ell_j-r}.\tag{6.28}$$

Thus we have

$$\widehat{gl}(\vec{\ell},\mathbb{A})_+(\mathbf{1})=\{0\}.\tag{6.29}$$

By a similar proof as that of Theorem 3.1,

$$U_\chi=U(\widehat{gl}(\vec{\ell},\mathbb{A})_-)\mathbf{1}.\tag{6.30}$$

Therefore, we have a Lie algebra module epimorphism $\nu:\mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A}))\to U_\chi$ defined by

$$\nu(u|0\rangle)=u\mathbf{1}\quad\text{for }u\in U(\widehat{gl}(\vec{\ell},\mathbb{A})_-).\tag{6.31}$$

Set

$$\ell=\min\{\ell_1,\ell_2,\ldots,\ell_n\}.\tag{6.32}$$

For $m\in\mathbb{N}+\ell+1$, we let

$$\tilde{gl}(\infty)_{(-)}^{(m)}=\mathrm{Span}\{\mathcal{E}_{-(l+1)n+i-1/2,-k-j+1/2}\mid i,j\in\overline{1,n},l,k\in\mathbb{N};l+k+\ell_j+1=m\}.\tag{6.33}$$

Then

$$\tilde{gl}(\infty)_{(-)}=\bigoplus_{m=\ell+1}^{\infty}\tilde{gl}(\infty)_{(-)}^{(m)}.\tag{6.34}$$

Moreover, we define

$$U_\chi^{(0)}=\mathbb{C}\mathbf{1},\qquad U_\chi^{(m)}=\{0\}\quad\text{for }m\in(-\mathbb{N}-1)\bigcup\overline{1,\ell}\tag{6.35}$$

and

$$U_\chi^{(m)}=\mathrm{Span}\left\{u_1u_2\cdots u_s\mathbf{1}\mid u_i\in\tilde{gl}(\infty)_{(-)}^{(m_i)};\sum_{i=1}^{s}m_i=m\right\}.\tag{6.36}$$

By (6.30),

$$U_\chi=\bigoplus_{m\in\mathbb{Z}}U_\chi^{(m)}\tag{6.37}$$

is a $\mathbb{Z}$-graded $\widehat{gl}(\vec{\ell},\mathbb{A})$-module. Furthermore, (6.33) implies the character

$$d(U_\chi,q)=\sum_{k=0}^{\infty}(\dim U_\chi^{(k)})q^k=\prod_{i=1}^{n}\prod_{r=1}^{\infty}\frac{1}{(1-q^{\ell_i+r})^{rn}}.\tag{6.38}$$

Therefore, (6.19) and (6.38) imply

$$\mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A}))\cong U_\chi.\tag{6.39}$$

Let λ be a linear function on $\mathcal{T}$ (cf. (3.13)) such that

$$\lambda(\kappa_0) = \chi, \qquad \lambda(\mathcal{E}_{l,-l}) = 0 \quad \text{for } l \in \mathcal{Z}. \tag{6.40}$$

Recall the Verma module M_λ defined in (3.64). Note that

$$U_\chi \cong M_\lambda \Bigg/ \left(\sum_{-1/2 \neq l \in \mathcal{Z}} U(\tilde{gl}(\infty)_-)(\mathcal{E}_{-l-1,l} \otimes v_\lambda) \right), \tag{6.41}$$

which is irreducible if $\chi \notin \mathbb{Z}$ by [17–19]. When $\chi \in \mathbb{N}$,

$$\begin{aligned} \bar{U}_\chi &= U(\tilde{gl}(\infty)_-)\mathcal{E}_{-1/2,-1/2}^{\chi+1}\mathbf{1} \\ &\cong \left(\sum_{l \in \mathcal{Z}} U(\tilde{gl}(\infty)_-)(\mathcal{E}_{-l-1,l}^{\lambda_l+1} \otimes v_\lambda) \right) \Bigg/ \left(\sum_{-1/2 \neq l \in \mathcal{Z}} U(\tilde{gl}(\infty)_-)(\mathcal{E}_{-l-1,l}^{\lambda_l+1} \otimes v_\lambda) \right) \end{aligned} \tag{6.42}$$

is the unique maximal proper submodule of U_χ (cf. (3.66)). Thus

$$U(\widehat{gl}(\vec{\ell}, \mathbb{A}))\nu^{-1}(\mathcal{E}_{-1/2,-1/2}^{\chi+1}\mathbf{1}) \tag{6.43}$$

is the unique maximal proper submodule of $\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$ (cf. (6.31)). When $n > 1$, (6.23) and (6.27) imply

$$\nu^{-1}(\mathcal{E}_{-1/2,-1/2}^{\chi+1}\mathbf{1}) = (t^{-1}\partial_t^{\ell_1} E_{n,1})^{\chi+1}|0\rangle. \quad \square \tag{6.44}$$

Next we want to present the definitions of vertex algebra and its module. For any two vector spaces U and W, we denote by $\mathrm{LM}(U, W)$ the set of all linear maps from U to W. Let z_1 and z_2 be two formal variables. We have the following convention of binomial expansions:

$$(z_1 - z_2)^r = \sum_{l=0}^{\infty} (-1)^l \binom{r}{l} z_1^{r-l} z_2^l \quad \text{for } r \in \mathbb{C}. \tag{6.45}$$

For a vector space V, we denote by

$$V[z^{-1}, z]] = \left\{ \sum_{i=m}^{\infty} v_i z^i \mid v_i \in V, m \in \mathbb{Z} \right\}, \tag{6.46}$$

the space of formal Laurent series with coefficients in V.

A *vertex algebra* is a vector space V with a linear map $Y(\cdot, z) : V \to \mathrm{LM}(V, V[z^{-1} : z]])$, an element $\partial \in \mathrm{End}\, V$ and an element $|0\rangle \in V$, satisfying the following conditions: given $u, v \in V$,

$$Y(|0\rangle, z) = \mathrm{Id}_V, \tag{6.47}$$

$$[\partial, Y(v, z)] = \frac{\mathrm{d}}{\mathrm{d}z} Y(v, z), \qquad Y(v, z)|0\rangle = e^{z\partial} v, \tag{6.48}$$

$$(z_1 - z_2)^m Y(u, z_1)Y(v, z_2) = (z_1 - z_2)^m Y(v, z_2)Y(u, z_1) \tag{6.49}$$

for some positive integer m. The above definition was proved in [12] to be equivalent to that of Borcherds [4].

An ideal U of a vertex algebra V is a subspace of V such that

$$\partial(U) \subset U, \qquad Y(v, z)U \subset U[z^{-1}, z]] \quad \text{for } v \in V. \tag{6.50}$$

A vertex algebra without proper nonzero ideals is called *simple*.

A module W of a vertex algebra $(V, Y(\cdot, z), |0\rangle, \partial)$ is a vector space with a linear map $Y_W(\cdot, z) : V \to \mathrm{LM}(W, W[z^{-1} : z]])$ such that given $u, v \in V$ and $w \in W$,

$$Y_W(|0\rangle, z) = \mathrm{Id}_W, \qquad Y_W(\partial v, z) = \frac{\mathrm{d}}{\mathrm{d}z} Y(v, z), \tag{6.51}$$

$$(z_1 - z_2)^m Y_W(u, z_1) Y_W(v, z_2) = (z_1 - z_2)^m Y_W(v, z_2) Y_W(u, z_1), \tag{6.52}$$
$$(z_0 + z_2)^m Y_W(u, z_0 + z_2) Y_W(v, z_2) w = (z_2 + z_0)^m Y_W(Y(u, z_0)v, z_2) w \tag{6.53}$$

for some positive integer m. This definition of module is equivalent to that in [14] (cf. [20,31]).

A *submodule* W_1 of W is a subspace such that

$$Y_W(v, z) W_1 \subset W_1[z^{-1}, z]] \quad \text{for } v \in V. \tag{6.54}$$

A module without proper nonzero submodule is called *irreducible*.

For a vector space U and any formal power series

$$f(z) = \sum_{r \in \mathbb{Z}} u_r z^{-r-1} \quad \text{with } u_r \in U, \tag{6.55}$$

we define

$$f(z)^+ = \sum_{r=0}^{\infty} u_r z^{-r-1}, \qquad f(z)^- = \sum_{r=0}^{\infty} u_{-r-1} z^r. \tag{6.56}$$

Now we define a linear transformation ∂ on $\widehat{gl}(\vec{\ell}, \mathbb{A})$ by

$$\partial(\kappa) = 0, \qquad \partial(t^m \partial_t^r E_{i,j}) = -m t^{m-1} \partial_t^r E_{i,j} \tag{6.57}$$

for $i, j \in \overline{1, n}$, $r \in \mathbb{N}$ and $m \in \mathbb{Z}$. Moreover, we define a linear transformation ∂ on $\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$ by

$$\partial(|0\rangle) = 0, \qquad \partial(u_1 u_2 \cdots u_s |0\rangle) = \sum_{i=1}^{s} u_1 \cdots u_{i-1} \partial(u_i) u_{i+1} \cdots u_s |0\rangle \tag{6.58}$$

for $u_i \in \widehat{gl}(\vec{\ell}, \mathbb{A})_-$. For $i, j \in \overline{1, n}$ and $r \in \mathbb{N} + \ell_j$, we denote

$$E_{i,j}(r, z) = \sum_{l \in \mathbb{Z}}^{\infty} t^l \partial_t^r E_{i,j} z^{-l-1}. \tag{6.59}$$

Furthermore, we define linear maps

$$Y^{\pm}(\cdot, z) : \widehat{gl}(\vec{\ell}, \mathbb{A})_- \to LM(\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})), \mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))[z^{-1}, z]]) \tag{6.60}$$

by

$$Y^{\pm}(t^{-m-1} \partial_t^r E_{i,j}, z) = \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}z^m} E_{i,j}(r, z)^{\pm}. \tag{6.61}$$

Now we define a linear map

$$Y(\cdot, z) : \mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})) \to LM(\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})), \mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))[z^{-1}, z]]) \tag{6.62}$$

by induction:

$$Y(|0\rangle, z) = \mathrm{Id}_{\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))}, \qquad Y(uv, z) = Y^-(u, z) Y(v, z) + Y(v, z) Y^+(u, z) \tag{6.63}$$

for $u \in \widehat{gl}(\vec{\ell}, \mathbb{A})_-$ and $v \in \mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$. In particular,

$$Y((t^{-m-1} \partial_t^r E_{i,j})|0\rangle, z) = \frac{1}{m!} \frac{\mathrm{d}^m}{\mathrm{d}z^m} E_{i,j}(r, z) \quad \text{for } i, j \in \overline{1, n}, r \in \mathbb{N} + \ell_j, m \in \mathbb{N}. \tag{6.64}$$

By Lemma 2.1 and the general theory of conformal algebras (cf. [22,33]), we have:

Theorem 6.2. *The family $(\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), \partial, |0\rangle)$ forms a vertex algebra. If $\chi \notin \mathbb{Z}$, the vertex algebra $(\mathcal{V}_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), \partial, |0\rangle)$ is simple. When $\chi \in \mathbb{N}$, the quotient space $V_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A}))$ forms a simple vertex algebra.*

If $\ell_i \in \{0,1\}$ for $i \in \overline{1,n}$, $\mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A}))$ with $\chi \in \mathbb{C}\setminus\mathbb{Z}$ and $V_\chi(\widehat{gl}(\vec{\ell},\mathbb{A}))$ with $\chi \in \mathbb{Z}$ are simple vertex operator algebras with the Virasoro element $-t^{-1}\partial_t(\sum_{i=1}^n E_{i,i})|0\rangle$.

Take constants $\chi, \iota \in \mathbb{C}$ such that $\chi \neq 0$. Let $\mathcal{M}$ be a weighted irreducible $\tilde{gl}(\infty)$-module satisfying (3.19) (also cf. (3.18)) and $\kappa_0|_{\mathcal{M}} = \chi \mathrm{Id}_{\mathcal{M}}$. For $i, j \in \overline{1,n}$ and $r \in \mathbb{N}+\ell_j$, we denote

$$E^\iota_{i,j}(r,z) = \sum_{l,k\in\mathbb{Z}} \langle \iota - k\rangle_r \mathcal{E}_{ln+i-1/2,kn-j+1/2} z^{-l-k-r-1} + r!\Im_{0,r}\delta_{i,j}\kappa_0 z^{-r-1} \tag{6.65}$$

if $\iota \notin \mathbb{Z}$, and

$$\begin{aligned} E^\iota_{i,j}(r,z) = \sum_{l,k=0}^{n} [&\langle -k-1\rangle_r \mathcal{E}_{(l-\iota)n+i-1/2,(k+\iota+1)n-j+1/2} z^{-l-k-\ell_i-r-2} \\ &+ \langle k+\ell_j\rangle_r \mathcal{E}_{(l-\iota)n+i-1/2,(\iota-k)n-j+/2} z^{-l+k+\ell_j-\ell_i-r-1} \\ &+ \langle k+\ell_j\rangle_r \mathcal{E}_{-(l+\iota+1)n+i-1/2,(\iota-k)n-j+1/2} z^{l+k+\ell_j-r} \\ &+ \langle -k-1\rangle_r \mathcal{E}_{-(l+\iota+1)n+i-1/2,(k+\iota+1)n-j+1/2} z^{l-k-r-1}] + \delta_{i,j} r!\Im_{0,r}\kappa_0 z^{-r-1} \end{aligned} \tag{6.66}$$

when $\iota \in \mathbb{Z}$ (cf. (2.26)). Furthermore, we define linear maps

$$Y^{\iota,\pm}_{\mathcal{M}}(\cdot,z) : \widehat{gl}(\vec{\ell},\mathbb{A})_- \to LM(\mathcal{M},\mathcal{M}[z^{-1},z]]) \tag{6.67}$$

by

$$Y^{\iota,\pm}_{\mathcal{M}}(t^{-m-1}\partial_t^r E_{i,j},z) = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m} E^\iota_{i,j}(r,z)^\pm. \tag{6.68}$$

Now we define a linear map

$$Y^\iota_{\mathcal{M}}(\cdot,z) : \mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A})) \to LM(\mathcal{M},\mathcal{M}[z^{-1},z]]) \tag{6.69}$$

by induction:

$$Y^\iota_{\mathcal{M}}(|0\rangle,z) = \mathrm{Id}_{\mathcal{M}}, \qquad Y(uv,z) = Y^{\iota,-}_{\mathcal{M}}(u,z)Y^\iota_{\mathcal{M}}(v,z) + Y^\iota_{\mathcal{M}}(v,z)Y^{\iota,+}_{\mathcal{M}}(u,z) \tag{6.70}$$

for $u \in \widehat{gl}(\vec{\ell},\mathbb{A})_-$ and $v \in \mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A}))$. By Theorems 3.1 and 3.2 and the general theory for vertex algebras (e.g. cf. Section 4.1 in [33]), we have:

Theorem 6.3. *The family $(\mathcal{M}, Y^\iota_{\mathcal{M}}(\cdot,z))$ forms an irreducible module of the vertex algebra $(\mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A})), Y(\cdot,z), \partial, |0\rangle)$.*

In the rest of this section, we want to show that certain familiar unitary highest-weight irreducible $\tilde{gl}(\infty)$-modules induce irreducible modules of the quotient simple vertex algebra $(V(\widehat{gl}(\vec{\ell},\mathbb{A})), Y(\cdot,z), \partial, |0\rangle)$ when $\chi \in \mathbb{Z}$. To this end, we need to use charged free fields.

First we use charged free fermionic field realization. Denote

$$\mathcal{Z}_+ = \mathbb{N}+1/2, \qquad \mathcal{Z}_- = -\mathcal{Z}_+. \tag{6.71}$$

Then

$$\mathcal{Z} = \mathcal{Z}_+ \bigcup \mathcal{Z}_-. \tag{6.72}$$

Let $\{\theta_l, \bar{\theta}_l \mid l \in \mathcal{Z}_-\}$ be a set of odd variables, that is,

$$\theta_l\theta_k = -\theta_k\theta_l, \qquad \theta_l\bar{\theta}_k = -\bar{\theta}_k\theta_l, \qquad \bar{\theta}_l\bar{\theta}_k = -\bar{\theta}_k\bar{\theta}_l. \tag{6.73}$$

Set

$$V_f = \mathbb{C}[\theta_l, \bar{\theta}_l \mid l \in \mathcal{Z}_-], \tag{6.74}$$

the polynomial algebra in odd variables $\{\theta_l, \bar{\theta}_l \mid l \in \mathcal{Z}_-\}$, where the subindex "$f$" stands for "fermionic". Moreover, we denote

$$\theta_l = \partial_{\bar{\theta}_{-l}}, \qquad \bar{\theta}_l = \partial_{\theta_{-l}} \quad \text{for } l \in \mathcal{Z}_+. \tag{6.75}$$

It can be verified that we have the following representation of $\tilde{gl}(\infty)$ on V_f:

$$\kappa_0|_{V_f} = \mathrm{Id}_{V_f}, \qquad \mathcal{E}_{l,k}|_{V_f} = \begin{cases} \bar{\theta}_l\theta_k & \text{if } l \in \mathcal{Z}_- \text{ or } -k \neq l \in \mathcal{Z}_+, \\ -\theta_k\bar{\theta}_l & \text{if } -k = l \in \mathcal{Z}_+ \end{cases} \tag{6.76}$$

for $l, k \in \mathcal{Z}$. Set

$$\partial = \sum_{l \in \mathcal{Z}} l\mathcal{E}_{-1-l,l}. \tag{6.77}$$

Given $i \in \overline{1,n}$, we define

$$\theta(\iota, i, z) = \sum_{l\in\mathbb{Z}} \theta_{ln-i+1/2} z^{\iota-l}, \qquad \bar{\theta}(\iota, i, z) = \sum_{l\in\mathbb{Z}} \bar{\theta}_{ln+i-1/2} z^{-\iota-l-1} \tag{6.78}$$

for $\iota \in \mathbb{C} \setminus \mathbb{Z}$, and

$$\theta(\iota, i, z) = \sum_{l=0}^{\infty} [\theta_{(-l+\iota)n-i+1/2} z^{\ell_i+l} + \theta_{(l+\iota+1)n-i+1/2} z^{-l-1}], \tag{6.79}$$

$$\bar{\theta}(\iota, i, z) = \sum_{l=0}^{\infty} [\bar{\theta}_{(l-\iota)n+i-1/2} z^{-\ell_i-l-1} + \bar{\theta}_{-(l+\iota+1)+i-1/2} z^{l}] \tag{6.80}$$

for $\iota \in \mathbb{Z}$. Then $\{\theta(\iota, i, z), \bar{\theta}(\iota, i, z) \mid i \in \overline{1,n}\}$ are *charged free fermionic fields*.

Set

$$\Theta = \sum_{l\in\mathcal{Z}_-} \mathbb{C}\theta_l, \qquad \bar{\Theta} = \sum_{l\in\mathcal{Z}_-} \mathbb{C}\bar{\theta}_l, \tag{6.81}$$

and

$$V_{f,0} = \mathbb{C} + \sum_{s=1}^{\infty} \bar{\Theta}^s \Theta^s, \qquad V_{f,r} = V_{f,0}\Theta^r, \qquad V_{f,-r} = \bar{\Theta}^r V_{f,0} \tag{6.82}$$

for $r \in \mathbb{N} + 1$. Then

$$V_f = \bigoplus_{k\in\mathbb{Z}} V_{f,k}. \tag{6.83}$$

We define a linear map $Y^\iota(\cdot, z) : V_f \to LM(V_f, V_f\{z\})$ by induction:

$$Y^\iota(1, z) = \mathrm{Id}_{V_f}, \tag{6.84}$$

$$\begin{aligned} Y^\iota(\theta_{-rn-i+1/2}u, z) \;=\; & \mathrm{Res}_{z_1} \sum_{s=0}^{\infty} (-1)^s \binom{-\iota}{s} z_1^{-\iota-s} z^{\iota} [(z_1 - z)^{s-r-\ell_i-1} \theta(\iota, i, z_1) Y^\iota(u, z) \\ & - (-1)^k (-z + z_1)^{s-r-\ell_i-1} Y^\iota(u, z)\theta(\iota, i, z_1)] \end{aligned} \tag{6.85}$$

and

$$\begin{aligned} Y^\iota(\bar{\theta}_{-(r+1)n+i-1/2}u, z) \;=\; & \mathrm{Res}_{z_1} \sum_{s=0}^{\infty} (-1)^s \binom{\iota}{s} z_1^{\iota-s} z^{-\iota} [(z_1 - z)^{s-r-1} \bar{\theta}(\iota, i, z_1) Y^\iota(u, z) \\ & - (-1)^k (-z + z_1)^{s-r-1} Y^\iota(u, z)\bar{\theta}(\iota, i, z_1)] \end{aligned} \tag{6.86}$$

for $i \in \overline{1,n}$, $r \in \mathbb{N}$ and $u \in V_{f,k}$. From Section 4.1 in [33], we have

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y^\iota(u,z_1)Y^\iota(v,z_2)-(-1)^{k_1k_2}z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y^\iota(v,z_2)Y^\iota(u,z_1)$$
$$=z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{k_1\iota}\delta\left(\frac{z_1-z_0}{z_2}\right)Y^\iota(Y^0(u,z_0)v,z_2) \tag{6.87}$$

for $u \in V_{f,k_1}$ and $v \in V_{f,k_2}$. In fact,

$$E^\iota_{i,j}(r,z)|_{V_f}=Y^\iota(\bar\theta_{-n+i-1/2}\theta_{-rn-j+1/2},z) \tag{6.88}$$

(cf. (6.65) and (6.66)),

$$Y^\iota(\theta_{-rn+i-1/2},z)=\frac{1}{(r+\ell_i)!}\frac{d^{r+\ell_i}}{z^{r+\ell_i}}\theta(\iota,i.z) \tag{6.89}$$

and

$$Y^\iota(\bar\theta_{-(r+1)n-i+1/2},z)=\frac{1}{r!}\frac{d^r}{z^r}\bar\theta(\iota,i.z) \tag{6.90}$$

for $i,j \in \overline{1,n}$.

Let $k \in \mathbb{N}+1$. Denote

$$v_0=1,\qquad v_k=\theta_{-1/2}\theta_{-3/2}\cdots\theta_{1/2-k},\qquad v_{-k}=\bar\theta_{-1/2}\bar\theta_{-3/2}\cdots\bar\theta_{1/2-k}. \tag{6.91}$$

Moreover, we define linear functions on $\mathcal{T}$ in (3.11):

$$\lambda^0(\kappa_0)=1,\qquad \lambda^0(\mathcal{E}_{l,-l})=0\quad\text{for } l\in\mathcal{Z}, \tag{6.92}$$
$$\lambda^{-k}(\kappa_0)=\lambda^{-k}(\mathcal{E}_{-r-1/2,r+1/2})=1,\qquad \lambda^{-k}(\mathcal{E}_{-l-1/2,l+1/2})=0 \tag{6.93}$$

for $r \in \overline{0,k-1}$ and $l \in \mathbb{Z}\setminus\overline{0,k-1}$, and

$$\lambda^k(\kappa_0)=-\lambda^k(\mathcal{E}_{r+1/2,-r-1/2})=1,\qquad \lambda^k(\mathcal{E}_{l+1/2,-l-1/2})=0 \tag{6.94}$$

for $r \in \overline{0,k-1}$ and $l \in \mathbb{Z}\setminus\overline{0,k-1}$. The following lemma is a straightforward consequence from the basic properties of free quadratic fields (or vertex operators associated with quadratic elements in the Fock space) (cf. [11,14,25]).

Lemma 6.4. *Each space $V_{f,k}$ forms a unitary irreducible highest-weight module of $\tilde{gl}(\infty)$ with the highest weight λ^k. Moreover, a linear function λ on $\mathcal{T}$ satisfies (3.60) and (3.66) and $\lambda(\kappa_0)=1$, if and only if λ is of the form λ^k for some $k\in\mathbb{Z}$. The family $(V_{f,0},Y^0(\cdot,z),1,\partial)$ is a simple vertex operator algebra isomorphic to $(V_1(\widehat{gl}(\vec\ell,\mathbb{A})),Y(\cdot,z),|0\rangle,\partial)$ in Theorem 6.2. Moreover, each $(V_{f,k},Y^\iota(|_{V_{f,0}},z)|_{V_{f,k}})$ is an irreducible $V_{f,0}$-module. The map $Y^\iota(|_{V_{f,l}},z)|_{V_{f,k}}$ is an intertwining operator of type $\left[{}_{V_{f,l}}\;{}^{V_{f,l+k}}\;{}_{V_{f,k}}\right]$.*

Now we want to deal with the high level case. Assume $1<\chi\in\mathbb{N}$. Form χth $\tilde{gl}(\infty)$-module tensor:

$$V_f^{\langle\chi\rangle}=V_f\otimes V_f\otimes\cdots\otimes V_f\quad(\chi\text{ copies}). \tag{6.95}$$

Then

$$\kappa_0|_{V_f^{\langle\chi\rangle}}=\chi\mathrm{Id}_{V_f^{\langle\chi\rangle}}. \tag{6.96}$$

Since V_f is a unitary $\tilde{gl}(\infty)$-module that is a direct sum of the highest-weight irreducible modules $V_{f,k}$ with weights λ^k satisfying (3.60), we can apply the tensor theory of modules of finite-dimensional general linear Lie algebras described Young tableaux to $V_f^{\langle\chi\rangle}$. Thus $V_f^{\langle\chi\rangle}$ is a completely reducible $\tilde{gl}(\infty)$-module. Moreover, for each weight λ satisfying (3.60) and (3.66) and $\lambda(\kappa_0)=\chi$, there exists a component of $V_f^{\langle\chi\rangle}$ that is a highest-weight irreducible $\tilde{gl}(\infty)$-module with weight λ.

Denote

$$1_\chi = 1 \otimes 1 \otimes \cdots \otimes 1 \quad (\chi \text{ copies}). \tag{6.97}$$

Since V_f is also a polynomial algebra in odd variables, $V_f^{\langle\chi\rangle}$ has an extended associative tensor algebra structure. Note the space

$$\bar{\Theta}\Theta = \sum_{l,k\in\mathcal{Z}_-} \mathbb{C}\bar{\theta}_l\theta_k. \tag{6.98}$$

We define the diagonal linear map $\varrho : \bar{\Theta}\Theta \to V_f^{\langle\chi\rangle}$ by

$$\varrho(\bar{\theta}_l\theta_k) = \sum_{i=1}^{\chi} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\bar{\theta}_l\theta_k} \otimes 1 \otimes \cdots \otimes 1 \tag{6.99}$$

for $l, k \in \mathcal{Z}_-$. Set

$$V_{f,0}^{[\chi]} = \mathbb{C}1_\chi + \sum_{r=1}^{\infty}[\varrho(\bar{\Theta}\Theta)]^r. \tag{6.100}$$

Moreover, we define the map

$$Y_\chi^\iota(\cdot, z) = Y^\iota(\cdot, z) \otimes Y^\iota(\cdot, z) \otimes \cdots \otimes Y^\iota(\cdot, z) \quad (\chi \text{ copies}) \tag{6.101}$$

and

$$\partial^{\langle\chi\rangle} = \sum_{i=1}^{\chi} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\partial} \otimes 1 \otimes \cdots \otimes 1 \tag{6.102}$$

(cf. (6.77)). Then the family $(V_{f,0}^{[\chi]}, Y_\chi^0(\cdot, z), 1_\chi, \partial^{\langle\chi\rangle})$ forms a simple vertex algebra isomorphic to $(V_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 6.2. For each irreducible $\tilde{gl}(\infty)$-module component U of $V_f^{\langle\chi\rangle}$, the family $(U, Y_\chi^\iota(|_{V_{f,0}^{[\chi]}}, z)|_U)$ forms an irreducible module of $(V_{f,0}^{[\chi]}, Y_\chi^0(\cdot, z), 1_\chi, \partial^{\langle\chi\rangle})$. The following lemma follows from the above facts and basic properties of vertex operator algebras (cf. [14,15]).

Lemma 6.5. *Suppose that χ is a positive integer and λ is a weight of $\tilde{gl}(\infty)$ satisfying (3.60) and (3.66) and $\lambda(\kappa_0) = \chi$. Let $\mathcal{M}$ be the irreducible highest-weight $\tilde{gl}(\infty)$-module with highest weight λ. Then the family $(\mathcal{M}, Y_{\mathcal{M}}^\iota)$ defined in (6.65)–(6.70) forms an irreducible module of the simple vertex algebra $(V_\chi(\widehat{gl}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 6.2, equivalently,*

$$Y_{\mathcal{M}}^\iota((t^{-1}\partial^{\ell_1}E_{n,1})^{\chi+1}|0\rangle, z) = 0 \tag{6.103}$$

when $n > 1$.

We remark that (6.103) can be proved easily by using the affine Lie algebra $\widehat{sl}(2, \mathbb{C})$ when $\ell_1 = \ell_n = 0$. Let $\mathcal{M}$ be the highest-weight irreducible module in Example 3.1. By the locality of $Y_{\mathcal{M}}^\iota(\cdot, z)$, (6.103) holds if and only if

$$Y_{\mathcal{M}}^\iota((t^{-1}\partial^{\ell_1}E_{n,1})^{\chi+1}|0\rangle, z)v_\lambda = 0. \tag{6.104}$$

According to the above lemma with $n = 2$, (6.104) holds if

$$\lambda(\mathcal{E}_{rn+1/2,-rn-1/2} - \mathcal{E}_{rn-1/2,-rn+1/2} + \delta_{r,0}\kappa_0) \in \mathbb{N} \quad \text{for } r \in \mathbb{Z} \tag{6.105}$$

and when $n > 2$

$$\lambda(\mathcal{E}_{rn+n-1/2,-rn-n+1/2} - \mathcal{E}_{rn+1/2,-rn-1/2} + \delta_{r,0}) \in \mathbb{N} \quad \text{for } r \in \mathbb{Z}, \tag{6.106}$$

Note that the condition of (6.105) and (6.106) is weaker than (3.66) when $n > 3$. Under the condition of (6.105) and (6.106), $\mathcal{M}$ may not be a unitary $\tilde{gl}(\infty)$-module. Therefore, we have proved the following main theorem:

Theorem 6.6. *Suppose that χ is a positive integer and λ is a weight of $\tilde{gl}(\infty)$ satisfying (3.60), (6.105), (6.106) and $\lambda(\kappa_0)=\chi$. Let $\mathcal{M}$ be the irreducible highest-weight $\tilde{gl}(\infty)$-module with highest weight λ. Then the family $(\mathcal{M}, Y^\iota_{\mathcal{M}})$ defined in (6.65)–(6.70) forms an irreducible module of the simple vertex algebra $(V_\chi(\widehat{gl}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ in Theorem 6.2.*

Next we use charged free bosonic field realization to study the case of negative integral χ. Set

$$V_b=\mathbb{C}[x_l,\bar{x}_l\mid l\in\mathcal{Z}_-], \tag{6.107}$$

the polynomial algebra in a set of ordinary commute variables $\{x_l,\bar{x}_l\mid l\in\mathcal{Z}_-\}$ (cf. (6.71)), where the subindex "b" stands for "bosonic". Moreover, we denote

$$x_l=\partial_{\bar{x}_{-l}},\qquad \bar{x}_l=-\partial_{x_{-l}}\quad\text{for } l\in\mathcal{Z}_+. \tag{6.108}$$

It can be verified that we have the following representation of $\tilde{gl}(\infty)$ on V_b:

$$\kappa_0|_{V_b}=-\mathrm{Id}_{V_b},\qquad \mathcal{E}_{-l,-k}|_{V_b}=-\partial_{x_l}\partial_{\bar{x}_k},\qquad \mathcal{E}_{l,-k}|_{V_b}=\bar{x}_l\partial_{\bar{x}_k}, \tag{6.109}$$

$$\mathcal{E}_{-l,k}|_{V_b}=-x_k\partial_{x_l},\qquad \mathcal{E}_{l,k}|_{V_b}=\bar{x}_lx_k \tag{6.110}$$

for $l,k\in\mathcal{Z}_-$. The operator ∂ acts on V_b by (6.77). Given $i\in\overline{1,n}$, we define

$$x(\iota,i,z)=\sum_{l\in\mathbb{Z}}x_{ln-i+1/2}z^{\iota-l},\qquad \bar{x}(\iota,i,z)=\sum_{l\in\mathbb{Z}}\bar{x}_{ln+i-1/2}z^{-\iota-l-1} \tag{6.111}$$

for $\iota\in\mathbb{C}\setminus\mathbb{Z}$, and

$$x(\iota,i,z)=\sum_{l=0}^{\infty}[x_{(-l+\iota)n-i+1/2}z^{\ell_i+l}+x_{(l+\iota+1)n-i+1/2}z^{-l-1}], \tag{6.112}$$

$$\bar{x}(\iota,i,z)=\sum_{l=0}^{\infty}[\bar{x}_{(l-\iota)n+i-1/2}z^{-\ell_i-l-1}+\bar{x}_{-(l+\iota+1)+i-1/2}z^{l}] \tag{6.113}$$

for $\iota\in\mathbb{Z}$. Then $\{x(\iota,i,z),\bar{x}(\iota,i,z)\mid i\in\overline{1,n}\}$ are *charged free bosonic fields*.

Set

$$X=\sum_{l\in\mathcal{Z}_-}\mathbb{C}x_l,\qquad \bar{X}=\sum_{l\in\mathcal{Z}_-}\mathbb{C}\bar{x}_l, \tag{6.114}$$

and

$$V_{b,0}=\mathbb{C}+\sum_{s=1}^{\infty}\bar{X}^sX^s,\qquad V_{b,r}=V_{b,0}X^r,\qquad V_{b,-r}=\bar{X}^rV_{b,0} \tag{6.115}$$

for $r\in\mathbb{N}+1$. Then

$$V_b=\bigoplus_{k\in\mathbb{Z}}V_{b,k}. \tag{6.116}$$

We define a linear map $Y^\iota(\cdot,z):V_b\to LM(V_b,V_b\{z\})$ by induction:

$$Y^\iota(1,z)=\mathrm{Id}_{V_b}, \tag{6.117}$$

$$\begin{aligned}Y^\iota(x_{-rn-i+1/2}u,z)\ =\ &\mathrm{Res}_{z_1}\sum_{s=0}^{\infty}(-1)^s\binom{-\iota}{s}z_1^{-\iota-s}z^\iota[(z_1-z)^{s-r-\ell_i-1}x(\iota,i,z_1)Y^\iota(u,z)\\ &-(-z+z_1)^{s-r-\ell_i-1}Y^\iota(u,z)x(\iota,i,z_1)]\end{aligned} \tag{6.118}$$

and

$$Y^\iota(\bar{x}_{-(r+1)n+i-1/2}u, z) = \mathrm{Res}_{z_1} \sum_{s=0}^{\infty} (-1)^s \binom{\iota}{s} z_1^{\iota-s} z^{-\iota}[(z_1-z)^{s-r-1}\bar{x}(\iota, i, z_1)Y^\iota(u,z) - (-z+z_1)^{s-r-1}Y^\iota(u,z)\bar{x}(\iota,i,z_1)] \tag{6.119}$$

for $i \in \overline{1,n}$, $r \in \mathbb{N}$ and $u \in V_{f,k}$. By Section 4.1 in [33], we have

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y^\iota(u,z_1)Y^\iota(v,z_2) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y^\iota(v,z_2)Y^\iota(u,z_1) = z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{k\iota}\delta\left(\frac{z_1-z_0}{z_2}\right)Y^\iota(Y^0(u,z_0)v,z_2) \tag{6.120}$$

for $u \in V_{b,k}$ and $v \in V_b$. In fact,

$$E^\iota_{i,j}(r,z)|_{V_b} = Y^\iota(\bar{x}_{-n+i-1/2}x_{-rn-j+1/2}, z) \tag{6.121}$$

(cf. (6.65) and (6.66))

$$Y^\iota(x_{-rn+i-1/2}, z) = \frac{1}{(r+\ell_i)!}\frac{d^{r+\ell_i}}{z^{r+\ell_i}}x(\iota, i.z) \tag{6.122}$$

and

$$Y^\iota(\bar{x}_{-(r+1)n-i+1/2}, z) = \frac{1}{r!}\frac{d^r}{z^r}\bar{x}(\iota, i.z) \tag{6.123}$$

for $i \in \overline{1,n}$.

Let $k \in \mathbb{N}+1$. Denote

$$v_0 = 1, \qquad v_k = x^k_{-1/2}, \qquad v_{-k} = \bar{x}^k_{-1/2}. \tag{6.124}$$

Moreover, we define linear functions on $\mathcal{T}$ in (3.13):

$$\lambda^0(\kappa_0) = -1, \qquad \lambda^0(\mathcal{E}_{l,-l}) = 0 \quad \text{for } l \in \mathcal{Z}, \tag{6.125}$$

$$\lambda^k(\kappa_0) = -1, \qquad \lambda^k(\mathcal{E}_{1/2,-1/2}) = -k, \qquad \lambda^k(E_{l+1/2,-l-1/2}) = 0 \tag{6.126}$$

for $0 \neq l \in \mathbb{Z}$, and

$$\lambda^{-k}(\kappa_0) = -1, \qquad \lambda^{-k}(\mathcal{E}_{-1/2,1/2}) = k, \qquad \lambda^{-k}(E_{l-1/2,-l+1/2}) = 0 \tag{6.127}$$

for $0 \neq l \in \mathbb{Z}$. The following lemma is a straightforward consequence from the basic properties of free quadratic fields (or vertex operators associated with quadratic elements in the Fock space) (cf. [11,14,25]).

Lemma 6.7. *Each space $V_{b,k}$ forms a unitary irreducible highest-weight module of $\tilde{gl}(\infty)$ with the highest weight λ^k. The family $(V_{b,0}, Y^0(\cdot,z), 1, \partial)$ is a simple vertex operator algebra isomorphic to $(V_{-1}(\widehat{gl}(\vec{\ell}, \mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ in Theorem 6.2. Moreover, each $(V_{b,k}, Y^\iota(|_{V_{b,0}}, z)|_{V_{b,k}})$ is an irreducible $V_{b,0}$-module. The map $Y^\iota(|_{V_{b,l}}, z)|_{V_{b,k}}$ is an intertwining operator of type $\begin{bmatrix} & V_{b,l+k} & \\ V_{b,l} & & V_{b,k}\end{bmatrix}$.*

Now we want to deal with the high level case. Assume $1 < \chi \in \mathbb{N}$. Form χth $\tilde{gl}(\infty)$-module tensor:

$$V_b^{\langle\chi\rangle} = V_b \otimes V_b \otimes \cdots \otimes V_b \quad (\chi \text{ copies}). \tag{6.128}$$

Then

$$\kappa_0|_{V_b^{\langle\chi\rangle}} = \chi \mathrm{Id}_{V_f^{\langle\chi\rangle}}. \tag{6.129}$$

Set

$$\mathcal{L} = \sum_{l,k \in \mathcal{Z}_-} \mathbb{C} x_l \partial_{x_k}, \qquad \bar{\mathcal{L}} = \sum_{l,k \in \mathcal{Z}_-} \mathbb{C} \bar{x}_l \partial_{\bar{x}_k}. \tag{6.130}$$

Then $\mathcal{L}$ and $\bar{\mathcal{L}}$ are Lie subalgebras of $\tilde{gl}(\infty)|_{V_b}$. Moreover, they are infinite-dimensional general Lie algebras. Denote

$$H = \sum_{l \in \mathcal{Z}_-} x_l \partial_{x_l}, \qquad \bar{H} = \sum_{l \in \mathcal{Z}_-} \bar{x}_l \partial_{\bar{x}_l}. \tag{6.131}$$

The subspace H is a toral Cartan subalgebra of $\mathcal{L}$ and the subspace $\bar{H}$ is a toral Cartan subalgebra of $\bar{\mathcal{L}}$. Moreover, X^k is an irreducible highest-weight $\mathcal{L}$-module with the highest weight λ_X^k determined by

$$\lambda_X^k(x_l \partial_{x_l}) = k\delta_{l,-1/2} \quad \text{for } l \in \mathcal{Z}_-, \tag{6.132}$$

and $\bar{X}^k$ is an irreducible highest-weight $\bar{\mathcal{L}}$-module with the highest weight λ_X^k determined by

$$\lambda_{\bar{X}}^k(\bar{x}_l \partial_{\bar{x}_l}) = k\delta_{l,-1/2} \quad \text{for } l \in \mathcal{Z}_-. \tag{6.133}$$

For a positive integer s, we define:

$$\Gamma_X^s = \{\lambda_X \in H^* \mid \lambda_X(x_{1/2-r}\partial_{x_{1/2-r}}) \in \mathbb{N}, \lambda_X(x_{-1/2-s-l}\partial_{x_{-1/2-s-l}}) = 0 \text{ for } r \in \overline{1,s}, l \in \mathbb{N}\} \tag{6.134}$$

and

$$\Gamma_{\bar{X}}^s = \{\lambda_{\bar{X}} \in \bar{H}^* \mid \lambda_{\bar{X}}(\bar{x}_{1/2-r}\partial_{\bar{x}_{1/2-r}}) \in \mathbb{N}, \lambda_{\bar{X}}(\bar{x}_{-1/2-s-l}\partial_{\bar{x}_{-1/2-s-l}}) = 0 \text{ for } r \in \overline{1,s}, l \in \mathbb{N}\}. \tag{6.135}$$

Set

$$\mathcal{C} = \mathbb{C}[x_l \mid l \in \mathcal{Z}_-] = \bigoplus_{r=0}^{\infty} X^r, \qquad \bar{\mathcal{C}} = \mathbb{C}[\bar{x}_l \mid l \in \mathcal{Z}_-] = \bigoplus_{r=0}^{\infty} \bar{X}^r. \tag{6.136}$$

Then

$$V_b = \bar{\mathcal{C}}\mathcal{C}. \tag{6.137}$$

By the tensor theory of modules of finite-dimensional general Lie algebras described Young tableaux, the s-tensor

$$\mathcal{C}^{\langle s\rangle} = \mathcal{C} \otimes \mathcal{C} \otimes \cdots \otimes \mathcal{C} \quad (s \text{ copies}) \tag{6.138}$$

can be decomposed as a direct sum of highest-weight irreducible $\mathcal{L}$-submodules, whose set of highest weights is exactly Γ_X^s. Similarly, the s-tensor

$$\bar{\mathcal{C}}^{\langle s\rangle} = \bar{\mathcal{C}} \otimes \bar{\mathcal{C}} \otimes \cdots \otimes \mathcal{C} \quad (s \text{ copies}) \tag{6.139}$$

can be decomposed as a direct sum of highest-weight irreducible $\mathcal{L}$-submodules, whose set of highest weights is exactly $\Gamma_{\bar{X}}^s$.

Since V_b is a unitary $\tilde{gl}(\infty)$-module, $V_b^{\langle\chi\rangle}$ is a completely reducible $\tilde{gl}(\infty)$-module. Let $s_1, s_2 \in \mathbb{N}$ such that $s_1 + s_2 = \chi$. Suppose that v_1 is a highest-weight vector of an irreducible component of the $\bar{\mathcal{L}}$-module $\bar{\mathcal{C}}^{\langle s\rangle}$ with $\lambda_{\bar{X}}$, and v_2 is a highest-weight vector of an irreducible component of the $\mathcal{L}$-module $\mathcal{C}^{\langle s\rangle}$ with λ_X. Then $v_1 \otimes v_2$ is the highest-weight vector of some irreducible component of the $\tilde{gl}(\infty)$-module $V_b^{\langle\chi\rangle}$ (cf. (6.128)), whose weight λ is given by:

$$\lambda(\kappa_0) = -\chi, \qquad \lambda(\mathcal{E}_{l,-l}) = \lambda_{\bar{X}}(\bar{x}_l \partial_{\bar{x}_l}), \qquad \lambda(\mathcal{E}_{-l,l}) = -\lambda_X(x_l \partial_{x_l}), \quad \text{for } l \in \mathcal{Z}_-. \tag{6.140}$$

Set

$$\mathcal{S}_\chi = \{\{3/2 - r, 5/2 - r, \ldots, (2\chi + 1)/2 - r\} \mid r \in \overline{1, \chi + 1}\}. \tag{6.141}$$

For $\lambda \in \mathcal{T}^*$, we define

$$\operatorname{supp}\lambda = \{l \in \mathcal{Z} \mid \lambda(\mathcal{E}_{l,-l}) \neq 0\}. \tag{6.142}$$

Set

$$\Gamma^{\chi} = \{\lambda \in \mathcal{T}^* \mid \lambda(\kappa_0) = -\chi, -s^{-1}|s|\lambda(\mathcal{E}_{s,-s}) \in \mathbb{N} \text{ for } s \in \mathcal{Z}; \operatorname{supp}\lambda \subset S \text{ for some } S \in \mathcal{S}_{\chi}\}. \tag{6.143}$$

Expression (6.140) implies that every element in Γ^{χ} is the highest weight of some irreducible component of the $\tilde{gl}(\infty)$-module $V_b^{\langle\chi\rangle}$.

Denote

$$1_{\chi} = 1 \otimes 1 \otimes \cdots \otimes 1 \quad (\chi \text{ copies}). \tag{6.144}$$

Since V_b is also a polynomial algebra, $V_b^{\langle\chi\rangle}$ has an extended commutative associative tensor algebra structure. Note the space

$$\bar{X}X = \sum_{l,k\in\mathcal{Z}_-} \mathbb{C}\bar{x}_l x_k. \tag{6.145}$$

We define the diagonal linear map $\varrho : \bar{X}X \to V_b^{\langle\chi\rangle}$ by

$$\varrho(\bar{x}_l x_k) = \sum_{i=1}^{\chi} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\bar{x}_l x_k} \otimes 1 \otimes \cdots \otimes 1 \tag{6.146}$$

for $l, k \in \mathcal{Z}_-$. Set

$$V_{b,0}^{[\chi]} = \mathbb{C}1_{\chi} + \sum_{r=1}^{\infty} [\varrho(\bar{X}X)]^r. \tag{6.147}$$

Moreover, we define the map

$$Y_{\chi}^{\iota}(\cdot, z) = Y^{\iota}(\cdot, z) \otimes Y^{\iota}(\cdot, z) \otimes \cdots \otimes Y^{\iota}(\cdot, z) \quad (\chi \text{ copies}) \tag{6.148}$$

and

$$\partial^{\langle\chi\rangle} = \sum_{i=1}^{\chi} 1 \otimes \cdots \otimes 1 \otimes \overset{i}{\partial} \otimes 1 \otimes \cdots \otimes 1 \tag{6.149}$$

(cf. (6.77)). Then the family $(V_{b,0}^{[\chi]}, Y_{\chi}^0(\cdot, z), 1_{\chi}, \partial_{(\iota)}^{\langle\chi\rangle})$ forms a simple vertex algebra isomorphic to $(V_{-\chi}(\widehat{gl}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 6.2. For each irreducible $\tilde{gl}(\infty)$-module component U of $V_f^{\langle\chi\rangle}$, the family $(U, Y_{\chi}^{\iota}(|_{V_{b,0}^{[\chi]}}, z)|_U)$ forms an irreducible module of $(V_{b,0}^{[\chi]}, Y_{\chi}^0(\cdot, z), 1_{\chi}, \partial^{\langle\chi\rangle})$. Therefore, we have proved the following main theorem:

Theorem 6.8. *Suppose that χ is a positive integer and $\lambda \in \Gamma^{\chi}$. Let $\mathcal{M}$ be the irreducible highest-weight $\tilde{gl}(\infty)$-module with highest weight λ. Then the family $(\mathcal{M}, Y_{\mathcal{M}}^{\iota})$ defined in (6.65)–(6.70) forms an irreducible module of the simple vertex algebra $(V_{-\chi}(\widehat{gl}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 6.2.*

7. Vacuum representations of $\hat{o}(\vec{\ell}, \mathbb{A})$ and $\widehat{sp}(\vec{\ell}, \mathbb{A})$

In this section, we study the vacuum representations of the Lie algebras $\hat{o}(\vec{\ell}, \mathbb{A})$ in (3.44) and $\widehat{sp}(\vec{\ell}, \mathbb{A})$ in (3.58), and their related vertex algebra structures. Their vertex algebra irreducible representations are investigated.

Recall the general settings in (6.1)–(6.9). Note that

$$\hat{o}(\vec{\ell}, \mathbb{A})_- = \sum_{i,j=1}^{n} \sum_{r=0}^{\infty} \sum_{m=1}^{\infty} \mathbb{C}(t^{-m}\partial_t^{r+\ell_j} E_{i,j} - (-1)^{\epsilon}(-\partial)^r t^{-m}\partial_t^{\ell_i} E_{j^*,i^*}) + \mathbb{C}\kappa \tag{7.1}$$

and

$$\hat{o}(\vec{\ell},\mathbb{A})_+ = \sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\mathbb{C}(t^m\partial_t^{r+\ell_j}E_{i,j}-(-1)^{\epsilon}(-\partial)^r t^m\partial_t^{\ell_i}E_{j^*,i^*})+\mathbb{C}\kappa. \tag{7.2}$$

The vacuum module

$$\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A})) = U(\hat{o}(\vec{\ell},\mathbb{A})_-)|0\rangle \tag{7.3}$$

and

$$\hat{o}(\vec{\ell},\mathbb{A})_+|0\rangle = \{0\}, \qquad \kappa(|0\rangle) = \chi|0\rangle, \tag{7.4}$$

where $\chi \in \mathbb{C}$.

Theorem 7.1. *The module $\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A}))$ is irreducible if $\chi \notin \mathbb{Z}$. When $\chi \in \mathbb{Z}$, the module $\mathcal{V}(\hat{o}(\vec{\ell},\mathbb{A}))$ has a unique maximal proper submodule $\bar{\mathcal{V}}_\chi(\hat{o}(\vec{\ell},\mathbb{A}))$, and the quotient*

$$V_\chi(\hat{o}(\vec{\ell},\mathbb{A})) = \mathcal{V}(\hat{o}(\vec{\ell},\mathbb{A}))/\bar{\mathcal{V}}_\chi(\hat{o}(\vec{\ell},\mathbb{A})) \tag{7.5}$$

is an irreducible $\hat{o}(\vec{\ell},\mathbb{A})$-module. Assume $\chi \in \mathbb{N}$. The submodule

$$\bar{\mathcal{V}}_\chi(\hat{o}(\vec{\ell},\mathbb{A})) = U(\hat{o}(\vec{\ell},\mathbb{A}))(t^{-1}\partial^{\ell_1}E_{n,1})^{\chi+1}|0\rangle \tag{7.6}$$

if $n>1$ and $\epsilon = 1$ (cf. (4.13)). When $\epsilon = 0$ and $n>3$, the submodule

$$\bar{\mathcal{V}}_\chi(\hat{o}(\vec{\ell},\mathbb{A})) = U(\hat{o}(\vec{\ell},\mathbb{A}))(t^{-1}(\partial^{\ell_1}E_{n-1,1}-\partial_t^{\ell_2}E_{n,2}))^{\chi+1}|0\rangle. \tag{7.7}$$

Proof. For $k \in \mathbb{Z}$, we define

$$\hat{o}(\vec{\ell},\mathbb{A})_{(k)} = \sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\mathbb{C}(t^{r-k}\partial_t^{r+\ell_j}E_{i,j}-(-1)^{\epsilon}(-\partial_t)^r t^{r-k}\partial_t^{\ell_i}E_{j^*,i^*})+\mathbb{C}\delta_{k,0}\kappa. \tag{7.8}$$

Then

$$\hat{o}(\vec{\ell},\mathbb{A}) = \bigoplus_{k\in\mathbb{Z}}\hat{o}(\vec{\ell},\mathbb{A})_{(k)} \tag{7.9}$$

is a $\mathbb{Z}$-graded Lie algebra by (2.60), (2.66) and (2.68) with $\iota = 0$. We remark that this grading is not conformal weight grading. Moreover, we define a $\mathbb{Z}$-grading on $\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A}))$ by

$$\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A}))_{(0)} = \mathbb{C}|0\rangle, \qquad \mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A}))_{(-m)} = \{0\} \quad \text{for } m \in \mathbb{N}+1 \tag{7.10}$$

and

$$\mathcal{V}_\chi(\hat{gl}(\vec{\ell},\mathbb{A}))_{(m)} = \mathrm{Span}\left\{u_1u_2\cdots u_s \mid u_i \in \hat{o}(\vec{\ell},\mathbb{A})_-\bigcap\hat{o}(\vec{\ell},\mathbb{A})_{(k_i)};\ \sum_{i=1}^{s}k_i = m\right\} \tag{7.11}$$

for $m \in \mathbb{N}+1$. Then

$$\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A})) = \bigoplus_{k\in\mathbb{Z}}\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A}))_{(k)} \tag{7.12}$$

is a $\mathbb{Z}$-graded $\hat{o}(\vec{\ell},\mathbb{A})$-module. Observe that

$$\left[\sum_{r,s\in\mathbb{N}}\mathbb{C}(t^{-s-1}\partial_t^{\ell_j+r}E_{i,j}-(-1)^{\epsilon}(-\partial_t)^r t^{-s-1}\partial_t^{\ell_i}E_{j^*,i^*})\right]\bigcap\hat{o}(\vec{\ell},\mathbb{A})_{(k)}$$
$$= \sum_{r=0}^{k-1}\mathbb{C}(t^{r-k}\partial^{r+\ell_j}E_{i,j}-(-1)^{\epsilon}(-\partial_t)^r t^{r-k}\partial_t^{\ell_j}E_{j^*,i^*}). \tag{7.13}$$

By [34],

$$\dim\left(\sum_{r=0}^{k-1}\mathbb{C}(t^{r-k}\partial^r-(-1)^\epsilon(-\partial_t)^r t^{r-k})\right)=k\epsilon+(-1)^\epsilon[|k/2|]. \tag{7.14}$$

Thus the character

$$d(\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A})),q)=\sum_{k=0}^{\infty}(\dim\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A}))_{(k)})q^k=\prod_{r=1}^{\infty}\frac{1}{(1-q^r)^{n(r(n-1)/2+r\epsilon+(-1)^\epsilon[|r/2|])}}. \tag{7.15}$$

Recall the Lie algebra $\tilde{gl}(\infty)$ defined in (3.10) and (3.11), and the Lie algebra $\mathcal{L}_0^{*,\vec{\ell}}$ in (5.12), which is a Lie subalgebra of $\tilde{gl}(\infty)$ by (5.11). Next we set

$$\mathcal{L}_{(-)}^{*,\vec{\ell}}=\sum_{i,j=1}^{n}\sum_{l,k=0}^{\infty}[\mathbb{C}((-1)^\epsilon\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} \\ -\langle -l-1\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2})] \tag{7.16}$$

and

$$\mathcal{L}_{(+)}^{*,\vec{\ell}}=\sum_{i,j=1}^{n}\sum_{l,k=0}^{\infty}[\mathbb{C}((-1)^\epsilon\langle -k-1\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,-kn-j+1/2} \\ -\langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,ln+i-1/2})+\mathbb{C}((-1)^\epsilon\langle k+\ell_j\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2} \\ -\langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2})]. \tag{7.17}$$

By (3.11), $\mathcal{L}_{(\pm)}^{*,\vec{\ell}}$ are Lie subalgebras of $\mathcal{L}_0^{*,\vec{\ell}}$ and

$$\mathcal{L}_0^{*,\vec{\ell}}=\mathcal{L}_{\iota,(-)}^{*,\vec{\ell}}+\mathcal{L}_{(+)}^{*,\vec{\ell}}+\mathbb{C}\kappa_0. \tag{7.18}$$

Recall the $\tilde{gl}(\infty)$-module U_χ defined in (6.20)–(6.24). Note that

$$\mathcal{L}_{(+)}^{*,\vec{\ell}}\mathbf{1}=\{0\} \tag{7.19}$$

because $\mathcal{L}_{(+)}^{*,\vec{\ell}}\subset\tilde{gl}(\infty)_{(+)}$. Thus

$$U_\chi^o=U(\mathcal{L}_{(-)}^{*,\vec{\ell}})\mathbf{1} \tag{7.20}$$

is an $\mathcal{L}_0^{*,\vec{\ell}}$-module, which satisfies the condition (5.45) with $s=\bar{\ell}+1$ (cf. (5.42)) and $\mathcal{M}_0=\mathbb{C}1\otimes\mathbf{1}$.

Note our notion (5.46). We have the $\hat{o}(\vec{\ell},\mathbb{A})$-module structure on U_χ^o defined by $\kappa=\chi\mathrm{Id}_{U_\chi^o}$ and

$$(E_{i,j})_{\vec{\ell}}^{*}(r,z)=\sum_{l,k=0}^{\infty}[\langle k\rangle_r((-1)^\epsilon\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} \\ -\langle -l-1\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2})z^{l+k-r}+\langle k\rangle_r((-1)^\epsilon\langle -k-1\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,-kn-j+1/2} \\ -\langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,ln+i-1/2})z^{-l+k-\ell_i-r-1} \\ +\langle -k-\ell_2-1\rangle_r((-1)^\epsilon\langle k+\ell_j\rangle_{\ell_j}\mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2} \\ -\langle -l-1\rangle_{\ell_i}\mathcal{E}_{(k+1)n-j+1/2,-(l+1)n+i-1/2})z^{l-k-\ell_j-r-1} \\ +\langle -k-\ell_j-1\rangle_r((-1)^\epsilon\langle k+\ell_j\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2} \\ -\langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2})z^{-l-k-\ell_i-\ell_j-r-2}] \tag{7.21}$$

for $i, j \in \overline{1,n}$ and $r \in \mathbb{N}$. In particular,

$$(E_{i,j})^*_{\vec{\ell}}(r,z)\mathbf{1} = \sum_{l,k=0}^{\infty} \langle k\rangle_r ((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} - \langle -l-1\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2}) z^{l+k-r}\mathbf{1}. \tag{7.22}$$

Thus we have

$$\hat{o}(\ell,\mathbb{A})_+(\mathbf{1}) = \{0\}. \tag{7.23}$$

By a similar proof as that of Theorem 3.1,

$$U^o_\chi = U(\hat{o}(\vec{\ell},\mathbb{A})_-)\mathbf{1}. \tag{7.24}$$

Therefore, we have a Lie algebra module epimorphism $\nu : \mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A})) \to U^o_\chi$ defined by

$$\nu(u|0\rangle) = u\mathbf{1} \quad \text{for } u \in U(\hat{o}(\vec{\ell},\mathbb{A})_-). \tag{7.25}$$

For $m \in \mathbb{N}+1$, we let

$$\mathcal{L}^{*,\vec{\ell}}_{(-),m} = \mathrm{Span}\{(-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} - \langle -l-1\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2} \mid i,j \in \overline{1,n}, l,k \in \mathbb{N}; l+k+1 = m\}. \tag{7.26}$$

Then

$$\mathcal{L}^{*,\vec{\ell}}_{(-)} = \bigoplus_{m=1}^{\infty} \mathcal{L}^{*,\vec{\ell}}_{(-),m}. \tag{7.27}$$

Moreover, we define

$$U^{o,(0)}_\chi = \mathbb{C}\mathbf{1}, \qquad U^{o,(m)}_\chi = \{0\} \quad \text{for } m \in (-\mathbb{N}-1) \tag{7.28}$$

and

$$U^{o,(m)}_\chi = \mathrm{Span}\left\{u_1u_2\cdots u_s\mathbf{1} \mid u_i \in \mathcal{L}^{*,\vec{\ell}}_{(-),m_i}; \sum_{i=1}^{s} m_i = m\right\}. \tag{7.29}$$

Expressions (7.8), (7.21) and (7.24) imply that

$$U^o_\chi = \bigoplus_{m\in\mathbb{Z}} U^{o,(m)}_\chi \tag{7.30}$$

is a $\mathbb{Z}$-graded $\hat{o}(\vec{\ell},\mathbb{A})$-module. Furthermore, (7.26) gives the character

$$d(U^o_\chi, q) = \sum_{m=0}^{\infty} (\dim U^{o,(m)}_\chi) z^m = \prod_{r=1}^{\infty} \frac{1}{(1-q^r)^{n(r(n-1)/2+r\epsilon+(-1)^{\epsilon}[|r/2|])}}. \tag{7.31}$$

Therefore, (7.15) and (7.31) yield

$$\mathcal{V}_\chi(\hat{o}(\vec{\ell},\mathbb{A})) \cong U^o_\chi. \tag{7.32}$$

Let λ be a linear function on $\mathcal{T}$ defined in (5.22) such that

$$\lambda(\kappa_0) = \chi, \qquad \lambda(\vartheta_l) = 0 \quad \text{for } l \in \mathbb{N} \tag{7.33}$$

(cf. (5.21)). Recall the Verma module M_λ defined in (5.39) with $\iota = 0$ and $\tau = *$. Note that

$$U^o_\chi \cong M_\lambda \Big/ \left(\sum_{l=1}^{\infty} U(\mathcal{L}^{*,\vec{\ell}}_{0,-}) f^*_{\epsilon,l} \otimes v_\lambda\right), \tag{7.34}$$

which is irreducible if $\chi \notin \mathbb{Z}$ by [17–19]. When $\chi \in \mathbb{N}$,

$$\bar{U}_{\chi}^{o} = U(\mathcal{L}_{0,-}^{*,\vec{\ell}})(f_{\epsilon,0}^{*})^{\chi+1}\mathbf{1} \tag{7.35}$$

is the unique maximal proper submodule of U_{χ}^{o}. Thus

$$U(\hat{o}(\vec{\ell},\mathbb{A}))\nu^{-1}((f_{\epsilon,0}^{*})^{\chi+1}\mathbf{1}) \tag{7.36}$$

is the unique maximal proper submodule of $\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))$. When $n>1$, (5.28), (7.21) and (7.25) imply

$$\nu^{-1}((f_{1,0}^{*})^{\chi+1}\mathbf{1}) = (t^{-1}\partial^{\ell_1}E_{n,1})^{\chi+1}|0\rangle. \tag{7.37}$$

Moreover, if $n>3$, (5.26), (7.21) and (7.25) yield

$$\nu^{-1}((f_{0,0}^{*})^{\chi+1}\mathbf{1}) = (t^{-1}(\partial^{\ell_1}E_{n-1,1}-\partial_t^{\ell_2}E_{n,2}))^{\chi+1}|0\rangle. \quad \square \tag{7.38}$$

Since $\hat{o}(\ell,\mathbb{A})$ is a Lie subalgebra of $\widehat{gl}(\vec{\ell},\mathbb{A})$, we view $\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))$ as a subspace of $\mathcal{V}_{\chi}(\widehat{gl}(\vec{\ell},\mathbb{A}))$. Recall the vertex algebra $(\mathcal{V}_{\chi}(\widehat{gl}(\vec{\ell},\mathbb{A})), Y(\cdot,z), \partial, |0\rangle)$ defined in (6.57)–(6.63). Then

$$(\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A})), Y(|_{\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))},z)|_{\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))}, \partial|_{\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))}, |0\rangle) \tag{7.39}$$

forms a vertex subalgebra. Let $\mathcal{M}$ be a weighted irreducible $\tilde{gl}(\infty)$-module satisfying (3.19) (also cf. (3.18)) and $\kappa_0|_{\mathcal{M}} = \chi\,\mathrm{Id}_{\mathcal{M}}$. Recall the linear map $Y_{\mathcal{M}}^{\iota}(\cdot,z)$ defined by (6.65)–(6.70). Now we obtain:

Theorem 7.2. *The family* (7.39) *forms a vertex algebra and* $(\mathcal{M}, Y_{\mathcal{M}}^{\iota}(|_{\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))},z))$ *forms an irreducible vertex algebra module of the vertex algebra* (7.39). *If* $\chi \notin \mathbb{Z}$, *the vertex algebra* $(\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A})), Y(\cdot,z), \partial, |0\rangle)$ *is simple. When* $\chi \in \mathbb{Z}$, *the quotient space* $V_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))$ *forms a simple vertex algebra. If* $\ell_i = \epsilon$ *for* $i \in \overline{1,n}$, $\mathcal{V}_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))$ *with* $\chi \in \mathbb{C}\setminus\mathbb{Z}$ *and* $V_{\chi}(\hat{o}(\vec{\ell},\mathbb{A}))$ *with* $\chi \in \mathbb{Z}$ *are simple vertex operator algebras with the Virasoro element*

$$\sum_{i=1}^{[|(n+1)/2|]} (-t^{-1}\partial_t E_{i,i} + (-1)^{\epsilon}(-\partial_t)^{\delta_{\epsilon,0}}t^{-1}\partial_t^{\epsilon}E_{i^*,i^*})|0\rangle. \tag{7.40}$$

Assume that χ is a positive integer. Recall the assumption (3.40) and the charged free fermionic field realization given in (6.71)–(6.86). We set

$$R^{o} = \mathrm{Span}\,\{\bar{\theta}_l\theta_k - (-1)^{\epsilon}\bar{\theta}_k\theta_l \mid l,k \in \mathcal{Z}_-\}. \tag{7.41}$$

Note the notion $V_f^{\langle\chi\rangle}$ defined in (6.95), the notion 1_{χ} defined in (6.97) and the map $\varrho : \bar{\Theta}\Theta \to V_f^{\langle\chi\rangle}$ defined by (6.81) and (6.99). Set

$$V_{\chi}^{o,f} = \mathbb{C}1_{\chi} + \sum_{r=1}^{\infty}[\varrho(R^{o})]^{r}. \tag{7.42}$$

In terms of (6.101) and (6.102), the family $(V_{\chi}^{o,f}, Y_{\chi}^{0}(\cdot,z), 1_{\chi}, \partial^{\langle\chi\rangle})$ forms a simple vertex algebra isomorphic to $(V_{\chi}(\hat{o}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ in Theorem 7.2. For each irreducible $\tilde{gl}(\infty)$-module component U of $V_f^{\langle\chi\rangle}$, the family $(U, Y_{\chi}^{\iota}(|_{V_{\chi}^{o,f}},z)|_U)$ forms an irreducible module of $(V_{\chi}^{o,f}, Y_{\chi}^{0}(\cdot,z), 1_{\chi}, \partial^{\langle\chi\rangle})$ when $\iota \notin \mathbb{Z}/2$.

Recall the charged free bosonic field realization given in (6.107)–(6.119). We set

$$R_o = \mathrm{Span}\,\{\bar{x}_l x_k - (-1)^{\epsilon}\bar{x}_k x_l \mid l,k \in \mathcal{Z}_-\}. \tag{7.43}$$

Note the notion $V_b^{\langle\chi\rangle}$ defined in (6.128), the notion 1_{χ} defined in (6.144) and the map $\varrho : \bar{\Theta}\Theta \to V_b^{\langle\chi\rangle}$ defined by (6.114) and (6.146). Set

$$V_{\chi}^{o,b} = \mathbb{C}1_{\chi} + \sum_{r=1}^{\infty}[\varrho(R_o)]^{r}. \tag{7.44}$$

According to (6.148) and (6.149), the family $(V_\chi^{o,b}, Y_\chi^0(\cdot,z), 1_\chi, \partial^{\langle\chi\rangle})$ forms a simple vertex algebra isomorphic to $(V_{-\chi}(\hat{o}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ in Theorem 7.2. For each irreducible $\tilde{gl}(\infty)$-module component U of $V_b^{\langle\chi\rangle}$, the family $(U, Y_\chi^\iota(|_{V_\chi^{o,b}}, z)|_U)$ forms an irreducible module of $(V_\chi^{o,b}, Y_\chi^0(\cdot,z), 1_\chi, \partial^{\langle\chi\rangle})$ when $\iota \notin \mathbb{Z}/2$. Thus we have:

Theorem 7.3. *Suppose that χ is a positive integer. Assume that λ is a weight of $\tilde{gl}(\infty)$ satisfying (3.60), (6.105), (6.106) and $\lambda(\kappa_0) = \chi$. Let $\mathcal{M}$ be the highest-weight irreducible $\tilde{gl}(\infty)$-module with highest weight λ. Then the family $(\mathcal{M}, Y_{\mathcal{M}}^\iota(\cdot,z))$ defined in (6.65)–(6.70) forms an irreducible module of the simple vertex algebra $(V_\chi(\hat{o}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ in Theorem 7.2 when $\iota \notin \mathbb{Z}/2$.*

If $\lambda \in \Gamma^\chi$ (cf. (6.143)) and $\mathcal{M}$ is the irreducible highest-weight $\tilde{gl}(\infty)$-module with highest weight λ, then the family $(\mathcal{M}, Y_{\mathcal{M}}^\iota(\cdot,z))$ defined in (6.65)–(6.70) forms an irreducible module of the simple vertex algebra $(V_{-\chi}(\hat{o}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ in Theorem 7.2 when $\iota \notin \mathbb{Z}/2$.

Assume $\iota \in \mathbb{Z}+1/2$. Recall the Lie algebra $\mathcal{L}_{\iota_0,\epsilon}^{*,\vec{m}}$ defined in (4.30) and the highest-weight irreducible $\mathcal{L}_{\iota_0,\epsilon}^{*,\vec{m}}$-module $\mathcal{M}_\lambda^{*,\epsilon}$ defined in (4.68) with $\tau = *$ and $\lambda(\kappa_0) = \chi$. By (4.75), we define operators

$$\begin{aligned} E_{i,j}^{\iota,*}(r,z) = \sum_{l,k\in\mathbb{Z}} &\langle k - m_j - \epsilon + 1/2\rangle_r ((-1)^\epsilon \langle k + m_j + \epsilon - 3/2\rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,kn-j+1/2} \\ &- \langle l + m_i + 1/2\rangle_{\ell_i} \mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2}) z^{-l-k-m_i-m_j-r-\epsilon-1} \\ &+ ((r+\ell_i)!\Im_{0,r+\ell_i} - r!\ell_i!\Im_{r,\ell_i})\delta_{i,j}\kappa_0 z^{-\ell_i-r-1} \end{aligned} \tag{7.45}$$

on $\mathcal{M}_\lambda^{*,\epsilon}$ for $i, j \in \overline{1,n}$ and $r \in \mathbb{N}$. For $l \in \mathbb{Z}$, we define

$$\varphi(l) = \begin{cases} -1 & \text{if } l - 1/2 + (m_{l_R} - \iota_0)n > 0, \\ 1 & \text{if } l - 1/2 + (m_{l_R} - \iota_0)n < 0 \end{cases} \tag{7.46}$$

(cf. (4.13) and (4.25)) by (4.16). Moreover, for $\lambda \in (\mathcal{T}^\epsilon)^*$ (cf. (4.47) and (4.59)), we define

$$\operatorname{supp}\lambda = \{l - 1/2 + (m_{l_R} - \iota_0)n \mid l \in \mathbb{N} - n_0\delta_{\epsilon,0}, \lambda(\vartheta_l^\epsilon) \neq 0\}. \tag{7.47}$$

Now we let

$$\Gamma_{\iota,\epsilon}^\chi = \{\lambda \in (\mathcal{T}^\epsilon)^* \mid \lambda(\kappa_0) = -\chi, \varphi(l)\lambda(\vartheta_l^\epsilon) \in \mathbb{N} \text{ for } l \in \mathbb{N} - n_0\delta_{\epsilon,0}; \operatorname{supp}\lambda \subset S \text{ for some } S \in \mathcal{S}_\chi\} \tag{7.48}$$

(cf. (6.141)).

Let $\iota \in \mathbb{Z}$. Recall the Lie algebra $\mathcal{L}_\iota^{*,\vec{\ell}}$ defined in (5.12) and the highest-weight irreducible $\mathcal{L}_\iota^{*,\vec{\ell}}$-module $\mathcal{M}_{\lambda^{*,\epsilon}}$ defined in (5.39) with $\tau = *$ and $\lambda^{*,\epsilon}(\kappa_0) = \chi$. By (5.48), we define operators

$$\begin{aligned} E_{i,j}^{\iota,*}(r,z) = \sum_{l,k=0}^\infty [&\langle k\rangle_r ((-1)^\epsilon \langle -k-1\rangle_{\ell_j} \mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2} \\ &- \langle -l-1\rangle_{\ell_i} \mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2}) z^{l+k-r} + \langle k\rangle_r ((-1)^\epsilon \langle -k-1\rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,-kn-j+1/2} \\ &- \langle l+\ell_i\rangle_{\ell_i} \mathcal{E}_{-kn-j+1/2,ln+i-1/2}) z^{-l+k-\ell_i-r-1} \\ &+ \langle -k-\ell_2-1\rangle_r ((-1)^\epsilon \langle k+\ell_j\rangle_{\ell_j} \mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2} \\ &- \langle -l-1\rangle_{\ell_i} \mathcal{E}_{(k+1)n-j+1/2,-(l+1)n+i-1/2}) z^{l-k-\ell_j-r-1} \\ &+ \langle -k-\ell_j-1\rangle_r ((-1)^\epsilon \langle k+\ell_j\rangle_{\ell_j} \mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2} \\ &- \langle l+\ell_i\rangle_{\ell_i} \mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2}) z^{-l-k-\ell_i-\ell_j-r-2} \\ &+ ((r+\ell_i)!\Im_{0,r+\ell_i} - (-1)^\epsilon r!\ell_i!\Im_{r,\ell_i})\delta_{i,j}\kappa_0 z^{-r-\ell_i-1}] \end{aligned} \tag{7.49}$$

on $\mathcal{M}_{\lambda^{*,\epsilon}}$ for $i, j \in \overline{1,n}$ and $r \in \mathbb{N}$. For $l \in \mathbb{N}$, we define

$$\bar{\varphi}(l) = \begin{cases} -1 & \text{if } l - 1/2 - \iota > 0, \\ 1 & \text{if } l - 1/2 - \iota < 0 \end{cases} \tag{7.50}$$

by (5.3). Moreover, for $\lambda \in \mathcal{T}^*$ (cf. (5.21) and (5.22)), we define

$$\mathrm{supp}\,\lambda = \{l - 1/2 - \iota \mid l \in \mathbb{N}, \lambda(\vartheta_l) \neq 0\}. \tag{7.51}$$

Now we let

$$\Gamma^{\chi}_{\iota,\epsilon} = \{\lambda \in \mathcal{T}^* \mid \lambda(\kappa_0) = -\chi, \bar{\varphi}(l)\lambda(\vartheta^{\epsilon}_l) \in \mathbb{N} \text{ for } l \in \mathbb{N}; \mathrm{supp}\,\lambda \subset S \text{ for some } S \in \mathcal{S}_\chi\} \tag{7.52}$$

(cf. (6.141)).

For convenience, we denote

$$\mathcal{M} = \begin{cases} \mathcal{M}^{*,\epsilon}_{\lambda} \text{ in (4.68)} & \text{if } \iota \in \mathbb{Z} + 1/2, \\ \mathcal{M}_{\lambda^{*,\epsilon}} \text{ in (5.39)} & \text{if } \iota \in \mathbb{Z}. \end{cases} \tag{7.53}$$

We define linear maps

$$Y^{\iota,\pm}_{\mathcal{M}}(\cdot, z) : \hat{o}(\vec{\ell}, \mathbb{A})_- \to LM(\mathcal{M}, \mathcal{M}[z^{-1}, z]]) \tag{7.54}$$

by

$$Y^{\iota,\pm}_{\mathcal{M}}(t^{-m-1}\partial_t^{r+\ell_j} E_{i,j} - (-1)^{\epsilon}(-\partial_t)^r t^{-m-1}\partial_t^{\ell_i} E_{j^*,i^*}, z) = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m} E^{\iota,*}_{i,j}(r, z)^{\pm} \tag{7.55}$$

for $i, j \in \overline{1, n}$ and $r, m \in \mathbb{N}$. Now we define a linear map

$$Y^{\iota}_{\mathcal{M}}(\cdot, z) : \mathcal{V}_\chi(\hat{o}(\vec{\ell}, \mathbb{A})) \to LM(\mathcal{M}, \mathcal{M}[z^{-1}, z]]) \tag{7.56}$$

by induction:

$$Y^{\iota}_{\mathcal{M}}(|0\rangle, z) = \mathrm{Id}_{\mathcal{M}}, \qquad Y(uv, z) = Y^{\iota,-}_{\mathcal{M}}(u, z)Y^{\iota}_{\mathcal{M}}(v, z) + Y^{\iota}_{\mathcal{M}}(v, z)Y^{\iota,+}_{\mathcal{M}}(u, z) \tag{7.57}$$

for $u \in \hat{o}(\vec{\ell}, \mathbb{A})_-$ and $v \in \mathcal{V}_\chi(\hat{o}(\vec{\ell}, \mathbb{A}))$.

By Theorems 4.2 and 5.2, the general theory for vertex algebras (e.g. cf. Section 4.1 in [33]), the charged free fermionic field realization and the charged free bosonic field realization, we obtain:

Theorem 7.4. *Assume $\iota \in \mathbb{Z}/2$. The family $(\mathcal{M}, Y^{\iota}_{\mathcal{M}}(\cdot, z))$ forms an irreducible module of the vertex algebra $(\mathcal{V}_\chi(\hat{o}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), \partial, |0\rangle)$. Suppose that χ is a positive integer. If (4.69) and (5.40) hold, then the family $(\mathcal{M}, Y^{\iota}_{\mathcal{M}}(\cdot, z))$ induces an irreducible module of the quotient simple vertex algebra $(V_\chi(\hat{o}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 7.2. When $\lambda \in \Gamma^{\chi}_{\iota,\epsilon}$ with $\iota \in \mathbb{Z}+1/2$ in (7.48) and $\lambda^{*,\epsilon} \in\in \Gamma^{\chi}_{\iota,\epsilon}$ with $\iota \in \mathbb{Z}$ in (7.52), the family $(\mathcal{M}, Y^{\iota}_{\mathcal{M}}(\cdot, z))$ induces an irreducible module of the quotient simple vertex algebra $(V_{-\chi}(\hat{o}(\vec{\ell}, \mathbb{A})), Y(\cdot, z), |0\rangle, \partial)$ in Theorem 7.2.*

Recall the general settings in (6.1)–(6.8). Observe that

$$\widehat{sp}(\vec{\ell}, \mathbb{A})_- = \sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\sum_{m=1}^{\infty} \mathbb{C}(t^{-m}\partial_t^{r+\ell_j} E_{i,j} - (-1)^{p(i)+p(j)+\epsilon}(-\partial)^r t^{-m}\partial_t^{\ell_i} E_{j^*,i^*}) + \mathbb{C}\kappa \tag{7.58}$$

and

$$\widehat{sp}(\vec{\ell}, \mathbb{A})_+ = \sum_{i,j=1}^{n}\sum_{r=0}^{\infty}\sum_{m=0}^{\infty} \mathbb{C}(t^{m}\partial_t^{r+\ell_j} E_{i,j} - (-1)^{p(i)+p(j)+\epsilon}(-\partial)^r t^{m}\partial_t^{\ell_i} E_{j^*,i^*}) + \mathbb{C}\kappa \tag{7.59}$$

(cf. (3.55)–(3.58)). The vacuum module

$$\mathcal{V}_\chi(\widehat{sp}(\vec{\ell}, \mathbb{A})) = U(\widehat{sp}(\vec{\ell}, \mathbb{A})_-)|0\rangle \tag{7.60}$$

and

$$\widehat{sp}(\vec{\ell}, \mathbb{A})_+|0\rangle = \{0\}, \qquad \kappa(|0\rangle) = \chi|0\rangle, \tag{7.61}$$

where $\chi \in \mathbb{C}$. By a similar proof as that of Theorem 7.1, we have:

Theorem 7.5. *The module $\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))$ is irreducible if $\chi \notin \mathbb{Z}$. When $\chi \in \mathbb{Z}$, the module $\mathcal{V}(\widehat{sp}(\vec{\ell},\mathbb{A}))$ has a unique maximal proper submodule $\bar{\mathcal{V}}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))$, and the quotient*

$$V_\chi(\widehat{sp}(\vec{\ell},\mathbb{A})) = \mathcal{V}(\widehat{sp}(\vec{\ell},\mathbb{A}))/\bar{\mathcal{V}}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A})) \tag{7.62}$$

is an irreducible $\widehat{sp}(\vec{\ell},\mathbb{A})$-module. Assume $\chi \in \mathbb{N}$. The submodule

$$\bar{\mathcal{V}}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A})) = U(\widehat{sp}(\vec{\ell},\mathbb{A}))(t^{-1}\partial^{\ell_1}E_{n,1})^{\chi+1}|0\rangle \tag{7.63}$$

if $n > 1$ and $\epsilon = 0$ (cf. (4.13)). When $\epsilon = 1$ and $n > 3$,

$$\bar{\mathcal{V}}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A})) = U(\widehat{sp}(\vec{\ell},\mathbb{A}))(t^{-1}(\partial^{\ell_1}E_{n-1,1} - \partial_t^{\ell_2}E_{n,2}))^{\chi+1}|0\rangle. \tag{7.64}$$

Since $\widehat{sp}(\ell,\mathbb{A})$ is a Lie subalgebra of $\widehat{gl}(\vec{\ell},\mathbb{A})$, we view $\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))$ as a subspace of $\mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A}))$. Recall the vertex algebra $(\mathcal{V}_\chi(\widehat{gl}(\vec{\ell},\mathbb{A})), Y(\cdot,z), \partial, |0\rangle)$ defined in (6.57)–(6.63). The family

$$(\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A})), Y(|_{\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))}, z)|_{\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))}, \partial|_{\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))}, |0\rangle) \tag{7.65}$$

forms a vertex subalgebra. Let $\mathcal{M}$ be a weighted irreducible $\tilde{gl}(\infty)$-module satisfying (3.19) (also cf. (3.18)) and $\kappa_0|_{\mathcal{M}} = \chi \mathrm{Id}_{\mathcal{M}}$. Recall the operator, the linear map $Y^\iota_{\mathcal{M}}(\cdot,z)$ defined by (6.65)–(6.70). Now we obtain

Theorem 7.6. *The family (7.65) forms a vertex algebra and $(\mathcal{M}, Y^\iota_{\mathcal{M}}(|_{\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))}, z))$ with $\iota \notin \mathbb{Z}/2$ forms an irreducible vertex algebra module of the vertex algebra (7.65). If $\chi \notin \mathbb{Z}$, the vertex algebra $(\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A})), Y(\cdot,z), \partial, |0\rangle)$ is simple. When $\chi \in \mathbb{Z}$, the quotient space $V_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))$ forms a simple vertex algebra. If $\ell_i = \epsilon$ for $i \in \overline{1,n}$, $\mathcal{V}_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))$ with $\chi \in \mathbb{C}\setminus\mathbb{Z}$ and $V_\chi(\widehat{sp}(\vec{\ell},\mathbb{A}))$ with $\chi \in \mathbb{Z}$ are simple vertex operator algebras with the Virasoro element*

$$\sum_{i=1}^{[|(n+1)/2|]} (-t^{-1}\partial_t E_{i,i} + (-1)^\epsilon(-\partial_t)^{\delta_{\epsilon,0}} t^{-1}\partial_t^\epsilon E_{i^*,i^*})|0\rangle. \tag{7.66}$$

Suppose that χ is a positive integer. Assume that λ is a weight of $\tilde{gl}(\infty)$ satisfying (3.60), (6.105), (6.106) and $\lambda(\kappa_0) = \chi$. Let $\mathcal{M}$ be the irreducible highest-weight $\tilde{gl}(\infty)$-module with highest weight λ. Then the family $(\mathcal{M}, Y^\iota_{\mathcal{M}}(\cdot,z))$ induces an irreducible module of the simple vertex algebra $(V_\chi(\widehat{sp}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ when $\iota \notin \mathbb{Z}/2$.

If $\lambda \in \Gamma^\chi$ (cf. (6.143)) and $\mathcal{M}$ is the highest-weight irreducible $\tilde{gl}(\infty)$-module with highest weight λ, then the family $(\mathcal{M}, Y^\iota_{\mathcal{M}}(\cdot,z))$ induces an irreducible module of the simple vertex algebra $(V_{-\chi}(\widehat{sp}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$ when $\iota \notin \mathbb{Z}/2$.

Assume $\iota \in \mathbb{Z}+1/2$. Recall the Lie algebra $\mathcal{L}^{\dagger,\vec{m}}_{\iota_0,\epsilon}$ defined in (4.31) and the highest-weight irreducible $\mathcal{L}^{\dagger,\vec{m}}_{\iota_0,\epsilon}$-module $\mathcal{M}^{\dagger,\epsilon}_\lambda$ defined in (4.68) with $\tau = \dagger$ and $\lambda(\kappa_0) = \chi$. By (4.76), we define operators

$$\begin{aligned} E^{\iota,\dagger}_{i,j}(r,z) = \sum_{l,k\in\mathbb{Z}} &\langle l - m_j - \epsilon + 1/2\rangle_r((-1)^\epsilon\langle k + m_j + \epsilon - 3/2\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,kn-j+1/2} \\ &- (-1)^{p(i)+p(j)}\langle l + m_i + 1/2\rangle_{\ell_i}\mathcal{E}_{(k+\epsilon-1)n-j+1/2,(l+1-\epsilon)n+i-1/2})z^{-l-k-m_i-m_j-r-\epsilon-1} \\ &+ ((r+\ell_i)!\Im_{0,r+\ell_i} - r!\ell_i!\Im_{r,\ell_i})\delta_{i,j}\kappa_0 z^{-\ell_i-r-1} \end{aligned} \tag{7.67}$$

on $\mathcal{M}^{\dagger,\epsilon}_\lambda$ for $i,j \in \overline{1,n}$ and $r \in \mathbb{N}$.

Let $\iota \in \mathbb{Z}$. Recall the Lie algebra $\mathcal{L}^{\dagger,\vec{\ell}}_\iota$ defined in (5.13) and the highest-weight irreducible $\mathcal{L}^{\dagger,\vec{\ell}}_\iota$-module $\mathcal{M}_{\lambda^{\dagger,\epsilon}}$ defined in (5.39) with $\tau = \dagger$ and $\lambda^{\dagger,\epsilon}(\kappa_0) = \chi$. By (5.49), we define operators

$$E^{\iota,\dagger}_{i,j}(r,z) = \sum_{l,k=0}^{\infty}[\langle k\rangle_r((-1)^\epsilon\langle -k-1\rangle_{\ell_j}\mathcal{E}_{(-l-1)n+i-1/2,-kn-j+1/2}$$

$$
\begin{aligned}
&- (-1)^{p(i)+p(j)}\langle -l-1\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,(-l-1)n+i-1/2)}z^{l+k-r}\\
&+ \langle k\rangle_r((-1)^{\epsilon}\langle -k-1\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,-kn-j+1/2} - \langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{-kn-j+1/2,ln+i-1/2})z^{-l+k-\ell_i-r-1}\\
&+ \langle -k-\ell_2-1\rangle_r((-1)^{\epsilon}\langle k+\ell_j\rangle_{\ell_j}\mathcal{E}_{-(l+1)n+i-1/2,(k+1)n-j+1/2}\\
&- \langle -l-1\rangle_{\ell_i}\mathcal{E}_{(k+1)n-j+1/2,-(l+1)n+i-1/2})z^{l-k-\ell_j-r-1}\\
&+ \langle -k-\ell_j-1\rangle_r((-1)^{\epsilon}\langle k+\ell_j\rangle_{\ell_j}\mathcal{E}_{ln+i-1/2,(k+1)n-j+1/2}\\
&- \langle l+\ell_i\rangle_{\ell_i}\mathcal{E}_{(k+1)n-j+1/2,ln+i-1/2})z^{-l-k-\ell_i-\ell_j-r-2}\\
&+ ((r+\ell_i)!\Im_{0,r+\ell_i} - (-1)^{\epsilon}r!\ell_i!\Im_{r,\ell_i})\delta_{i,j}\kappa_0 z^{-r-\ell_i-1}]
\end{aligned}
\tag{7.68}
$$

on $\mathcal{M}_{\lambda^{\dagger,\epsilon}}$ for $i, j \in \overline{1,n}$ and $r \in \mathbb{N}$.

For convenience, we denote

$$
\mathcal{M} = \begin{cases} \mathcal{M}_{\lambda}^{\dagger,\epsilon} & \text{in (4.68)} \quad \text{if } \iota \in \mathbb{Z}+1/2,\\ \mathcal{M}_{\lambda^{\dagger,\epsilon}} & \text{in (5.39)} \quad \text{if } \iota \in \mathbb{Z}. \end{cases}
\tag{7.69}
$$

We define linear maps

$$
Y_{\mathcal{M}}^{\iota,\pm}(\cdot,z) : \widehat{sp}(\vec{\ell},\mathbb{A})_- \to LM(\mathcal{M},\mathcal{M}[z^{-1},z]])
\tag{7.70}
$$

by

$$
Y_{\mathcal{M}}^{\iota,\pm}(t^{-m-1}\partial_t^{r+\ell_j}E_{i,j} - (-1)^{p(i)+p(j)}\epsilon(-\partial_t)^r t^{-m-1}\partial_t^{\ell_i}E_{j^*,i^*},z) = \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}E_{i,j}^{\iota,\dagger}(r,z)^{\pm}
\tag{7.71}
$$

for $i, j \in \overline{1,n}$ and $r, m \in \mathbb{N}$. Now we define a linear map

$$
Y_{\mathcal{M}}^{\iota}(\cdot,z) : \mathcal{V}_{\chi}(\widehat{sp}(\vec{\ell},\mathbb{A})) \to LM(\mathcal{M},\mathcal{M}[z^{-1},z]])
\tag{7.72}
$$

by induction:

$$
Y_{\mathcal{M}}^{\iota}(|0\rangle,z) = \mathrm{Id}_{\mathcal{M}}, \qquad Y(uv,z) = Y_{\mathcal{M}}^{\iota,-}(u,z)Y_{\mathcal{M}}^{\iota}(v,z) + Y_{\mathcal{M}}^{\iota}(v,z)Y_{\mathcal{M}}^{\iota,+}(u,z),
\tag{7.73}
$$

for $u \in \widehat{sp}(\vec{\ell},\mathbb{A})_-$ and $v \in \mathcal{V}_{\chi}(\widehat{sp}(\vec{\ell},\mathbb{A}))$.

By Theorems 4.2 and 5.2, the general theory for vertex algebras (e.g. cf. Section 4.1 in [33]), the charged free fermionic field realization and the charged free bosonic field realization, we obtain:

Theorem 7.7. *Assume $\iota \in \mathbb{Z}/2$. The family $(\mathcal{M}, Y_{\mathcal{M}}^{\iota}(\cdot,z))$ forms an irreducible module of the vertex algebra $(\mathcal{V}_{\chi}(\widehat{sp}(\vec{\ell},\mathbb{A})), Y(\cdot,z), \partial, |0\rangle)$. Suppose that χ is a positive integer. If (4.69) and (5.40) hold, then the family $(\mathcal{M}, Y_{\mathcal{M}}^{\iota}(\cdot,z))$ induces an irreducible module of the quotient simple vertex algebra $(V_{\chi}(\widehat{sp}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$. When $\lambda \in \Gamma_{\iota,\epsilon}^{\chi}$ with $\iota \in \mathbb{Z}+1/2$ in (7.48) and $\lambda^{\dagger,\epsilon} \in\in \Gamma_{\iota,\epsilon}^{\chi}$ with $\iota \in \mathbb{Z}$ in (7.52), the family $(\mathcal{M}, Y_{\mathcal{M}}^{\iota}(\cdot,z))$ induces an irreducible module of the quotient simple vertex algebra $(V_{-\chi}(\widehat{sp}(\vec{\ell},\mathbb{A})), Y(\cdot,z), |0\rangle, \partial)$.*

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